The Standard Model of Particle Physics

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Kapitel 2

The Four Pillars of the Standard Model

In our fundamental research of elementary particle physics we are guided by the endeavour to find answers to the fundamental questions of the Universe:

* What is it made of?
* How has it developped?
* Which are the building blocks of matter and which forces keep them together?

The status as of today is that we are able to

* break down known matter to a few fundamental particles;
* trace back the various interactions to fundamental forces between the particles;
* describe the physical laws mathematically with simple fundamental principles.

The Standard Model (SM) of Particle Physics, developed in the early 1970s, summarizes the today known fundamental structures of matter and forces (except for gravity). It is the result of theories and discoveries developed and made since the 1930s. The SM has been able to explain almost all experimental results and furthermore it precisely predicted a wide variety of phenomena. With precision experiments performed at previous and current colliders, the SM has been established as a physics theory tested to highest precision at the quantum level. With the discovery of the Higgs boson we now have a consistent mathematical framework to describe physics all the way up to the Planck scale.

The SM is based on four pillars:

* The local gauge symmetry group $SU(3)_C \times SU(2)_L \times U(1)_Y$.
* Its particle content with 12 matter particles and the gauge/interaction particles of the fundamental forces.
* The fundamental forces given by the strong and the electroweak force.
* The Higgs mechanism for the generation of particle masses without violating gauge invariance.

Kapitel 3

On our Way to the SM - Gauge Symmetries

Quantum field theory (QFT) provides the mathematical framework to describe elementary particles and their interactions. In QFT, we combine the principles of classical field theory and quantum mechanics. The Lagrangian of the QFT controls the dynamics and kinematics of the theory. Particles are described in terms of a dynamical field. In order to construct the Lagrangian of a theory, namely of the Standard Model, we proceed as follows: After first postulating the *set of symmetries* of the system, the most general *renormalizable* Lagrangian is constructed from the particle/field content of the system, that fulfills these symmetries.

The Lagrangian for all relativistic quantum field theories has to observe the global Poincaré symmetry. Poincaré transformations P in Minokowski space consist of a Lorentz transformation Λ^μ_ν and a translation by a^μ,

$$P = \{x^\mu \to x^{'\mu} = \Lambda^\mu_\nu x^\nu + a^\mu \ : \ \Lambda^\mu_\nu \in L, a^\mu \in \mathbb{R}^4\} \ . \tag{3.1}$$

Local gauge symmetry is an *internal* symmetry of the Lagrangian. We know such a gauge theory already, namely Maxwell's theory of electromagnetic interactions. The fact that we have the freedom to choose many potentials that describe the same electromagnetic fields is called *gauge invariance*. The gauge invariance of electromagnetism can be phrased in terms of a continuous symmetry of the Lagrangian. It leads through *Noether's theorem*[1] to the conservation of the electric charge and other important consequences. *Noether's theorem* states that for each symmetry of the action integral with respect to a continuous transformation exists a conservation law that can be derived from the Lagrangian. In accordance with the principle of local action we require the symmetry transformations of the fields rather to be local than global. We will see that this implies the *gauge principle*.

The *gauge principle* is the procedure to obtain an interaction term from a free Lagrangian that is symmetric with respect to a continous symmetry: When the global symmetry group is made *local* this has to be accompanied by the inclusion of additional fields with kinetic and *interaction* terms in the action in such a way that the resulting extended Lagrangian is covariant (*i.e.* the form of the physical laws does not change) with respect to a new extended group of local transformations.

[1] Emmi Noether was an German mathematician (1882-1935) who made fundamental contributions to abstract algebra and theoretical physics.

Let us look at the example of quantum electrodynamics. We start with the Dirac Lagrangian of a free fermion field Ψ with mass m. It is given by

$$\mathcal{L}_0 = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \; . \tag{3.2}$$

It is symmetric with respect to a global $U(1)$ transformation given by

$$\Psi(x) \to \exp(-i\alpha)\Psi(x) = \Psi - i\alpha\Psi + \mathcal{O}(\alpha^2) \; . \tag{3.3}$$

And for the adjoint spinor $\bar{\Psi} = \Psi^\dagger \gamma^0$ we have

$$\bar{\Psi}(x) \to \exp(i\alpha)\bar{\Psi}(x) \; . \tag{3.4}$$

The Noether current associated with this symmetry is

$$j^\mu = \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \Psi)} \frac{\delta \Psi}{\delta \alpha} + \frac{\delta \bar{\Psi}}{\delta \alpha} \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \bar{\Psi})} = i\bar{\Psi}\gamma^\mu(-i\Psi) = \bar{\Psi}\gamma^\mu \Psi \; , \tag{3.5}$$

where

$$\partial_\mu j^\mu = 0 \; . \tag{3.6}$$

When we also consider the coupling to a photon, the Lagrangian reads

$$\mathcal{L} = \bar{\Psi}\gamma^\mu(i\partial_\mu - qA_\mu)\Psi - m\bar{\Psi}\Psi = \mathcal{L}_0 - qj^\mu A_\mu \; , \tag{3.7}$$

with j^μ given in Eq. (3.5). The kinetic Lagrangian of the photon field

$$\mathcal{L}_{kin} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad \text{with} \qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{3.8}$$

is invariant under the local gauge transformation of the external photon field A_μ,

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x) \; . \tag{3.9}$$

When we apply this gauge transformation to the Lagrangian Eq. (3.7) it becomes

$$\mathcal{L} \to \mathcal{L} = \mathcal{L}_0 - qj^\mu A_\mu - \underbrace{qj^\mu \partial_\mu \Lambda}_{q\bar{\Psi}\gamma^\mu\Psi\partial_\mu\Lambda} \; . \tag{3.10}$$

This means that $\mathcal{L}$ is not gauge invariant. The gauge transformations of the fields Ψ and $\bar{\Psi}$ must be changed in such a way that the Lagrangian becomes gauge invariant. This is done by localizing the transformation Eq. (3.3), *i.e.* by introducing an x-dependent parameter α, hence $\alpha = \alpha(x)$. By this we obtain

$$i\partial_\mu \Psi \to i\exp(-i\alpha)\partial_\mu \Psi + \exp(-i\alpha)\Psi(\partial_\mu \alpha) \; , \tag{3.11}$$

so that

$$\mathcal{L}_0 \to \mathcal{L}_0 + \bar{\Psi}\gamma^\mu \Psi \partial_\mu \alpha \; . \tag{3.12}$$

This term cancels the additional term in Eq. (3.10) if

$$\alpha(x) = q\Lambda(x) \; . \tag{3.13}$$

Thereby the complete local gauge transformation is given by

$$\Psi \rightarrow \Psi'(x) = U(x)\Psi(x) \qquad \text{with} \qquad U(x) = exp(-iq\Lambda(x)) \qquad (U \text{ unitary}) \tag{3.14}$$

$$\bar{\Psi} \rightarrow \bar{\Psi}'(x) = \bar{\Psi}(x)U^\dagger(x) \tag{3.15}$$

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu\Lambda(x) = U(x)A_\mu(x)U^\dagger(x) - \frac{i}{q}U(x)\partial_\mu U^\dagger(x) \, . \tag{3.16}$$

The Lagrangian is tranformed as ($U^\dagger = U^{-1}$)

$$\begin{aligned} \mathcal{L} \rightarrow \mathcal{L}' &= \bar{\Psi}\gamma^\mu U^{-1} i\partial_\mu(U\Psi) - q\bar{\Psi}U^{-1}\gamma^\mu \left(UA_\mu U^{-1} - \frac{i}{q}U\partial_\mu U^{-1} \right) U\Psi - m\bar{\Psi}U^{-1}U\Psi \\ &= \bar{\Psi}\gamma^\mu i\partial_\mu\Psi + \bar{\Psi}\gamma^\mu(U^{-1}i(\partial_\mu U))\Psi - q\bar{\Psi}\gamma^\mu\Psi A_\mu + \bar{\Psi}\gamma^\mu(i(\partial_\mu U^{-1})U)\Psi - m\bar{\Psi}\Psi \\ &= \mathcal{L} + i\bar{\Psi}\gamma^\mu\partial_\mu(\underbrace{U^{-1}U}_{\mathbb{1}})\Psi = \mathcal{L} \, . \end{aligned} \tag{3.17}$$

Minimal substitution $p_\mu \rightarrow p_\mu - qA_\mu$ leads to

$$i\partial_\mu \rightarrow i\partial_\mu - qA_\mu \equiv iD_\mu \, . \tag{3.18}$$

Here $D_\mu(x)$ is the *covariant derivative*. The expression *covariant* means, it transforms as the field does,

$$\Psi(x) \rightarrow U(x)\Psi(x) \qquad \text{and} \qquad D_\mu\Psi(x) \rightarrow U(x)(D_\mu\Psi(x)) \, . \tag{3.19}$$

This implies

$$(D_\mu\Psi)' = D'_\mu\Psi' = D'_\mu U\Psi \stackrel{!}{=} UD_\mu\Psi \, , \tag{3.20}$$

so that the covariant derivative transforms as

$$D'_\mu = UD_\mu U^{-1} = \exp(-iq\Lambda)(\partial_\mu + iqA_\mu)\exp(iq\Lambda) = \partial_\mu + iq\partial_\mu\Lambda + iqA_\mu \stackrel{(3.9)}{=} \partial_\mu + iqA'_\mu \, . \tag{3.21}$$

Thereby

$$\mathcal{L} = \bar{\Psi}\gamma^\mu iD_\mu\Psi - m\bar{\Psi}\Psi \tag{3.22}$$

is obviously gauge invariant.

As stated above the kinetic Lagrangian of the photons is given by

$$\mathcal{L}_{kin} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad \text{with} \qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \, . \tag{3.23}$$

The field strength tensor $F^{\mu\nu}$ can be expressed in terms of the covariant derivative (verify!): We choose the following ansatz for the tensor of rank 2,

$$[D_\mu, D_\nu] = [\partial_\mu + iqA_\mu, \partial_\nu + iqA_\nu] = iq[\partial_\mu, A_\nu] + iq[A_\mu, \partial_\nu] = iq(\partial_\mu A_\nu - \partial_\nu A_\mu) \, . \tag{3.24}$$

Thereby we have for the field strength tensor

$$F^{\mu\nu} = \frac{-i}{q}[D^\mu, D^\nu] \, . \tag{3.25}$$

It behaviour under transformation is given by

$$\frac{-i}{q}[UD^\mu U^{-1}, UD^\nu U^{-1}] = \frac{-i}{q}U[D^\mu, D^\nu]U^{-1} = UF^{\mu\nu}U^{-1} \, . \tag{3.26}$$

3.1 Non-Abelian Gauge Groups

We now extend the considerations about local gauge invariance to gauge groups that are more complicated than the group of phase rotations. We will see that it is possible to have local gauge invariance by following the same strategy as the one applied for electrodynamics. Apart from being more complex, the basic difference will be the emergence of interactions among the gauge bosons as consequence of the non-Abelian nature of the gauge symmetry.

The motivation of introducing the more complicated structure is given *e.g.* by the fact that the near degeneracy of the neutron and proton masses, the charge-independence of the nuclear forces and subsequent observations suggested the notion of *isospin* conservation in the strong interactions. This means that the physics laws should be invariant under rotations in isospin space and hence the proton and neutron should appear symmetrically in all equations. In other words, if electromagnetism can be neglected the isospin orientation is of no significance and thereby the distinction between proton and neutron becomes entirely a matter of arbitrary convention. The existence of two distinct kinds of nucleons could then be inferred from the properties of the ground state of the ^{4}He nucleus. The free-nucleon Lagrangian would then be given by

$$\mathcal{L}_0 = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \tag{3.27}$$

in terms of the composite fermion fields

$$\Psi = \begin{pmatrix} p \\ n \end{pmatrix} . \tag{3.28}$$

We generalize this to N Dirac fields ψ_i of mass m. We have the Lagrangian

$$\mathcal{L} = \sum_{j=1...N} \bar{\psi}_j i\gamma^\mu \partial_\mu \psi_j - m \sum_{j=1...N} \bar{\psi}_j \psi_j . \tag{3.29}$$

The Lagrangian is symmetric under $U(N)$ where $U(N)$ is the group of unitary $N \times N$ matrices. We consider the following transformation

$$\psi_j \to \sum_{k=1...N} U_{jk}\psi_k \equiv U_{jk}\psi_k , \tag{3.30}$$

where we have used in the last equation the Einstein sum convention, *i.e.* that we sum over repeated indices. We hence have

$$\Psi \to U\Psi \qquad \text{with} \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} , \quad \text{hence} \quad \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} \to \begin{pmatrix} U_{1k}\psi_k \\ U_{2k}\psi_k \\ \vdots \\ U_{Nk}\psi_k \end{pmatrix} \tag{3.31}$$

and

$$\mathcal{L} = \bar{\Psi} i\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi \to \bar{\Psi}U^{-1} i\gamma^\mu \partial_\mu U\Psi - m\bar{\Psi}U^{-1}U\Psi = \mathcal{L} . \tag{3.32}$$

Examples are:

- $\Psi = \begin{pmatrix} p \\ n \end{pmatrix}$: $SU(2)$-transformations in isospin space, proton-neutron doublet.

- $\Psi = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$: $SU_L(2)$, weak interaction acting on left-handed fermions.

- $\Psi = (q_1, q_2, q_3)^T$, quarks, $SU(3)$. Here each q_i $(i = 1, 2, 3)$ is a four-component spinor. The Lagrangian is invariant under $SU(3)$ transformations.

3.2 The Matrices of the $SU(N)$

We now consider the matrices of the special unitary group $SU(N)$ which consists of the unitary $N \times N$ matrices with complex entries and determinant $+1$. It is a subgroup of the $U(N)$. The elements of the $SU(N)$ can be represented through

$$U = \exp\left(i\theta^a \frac{\lambda^a}{2}\right) \qquad \text{with} \qquad \theta^a \in \mathbb{R} \; . \tag{3.33}$$

The $\lambda^a/2$ are the generators of the group $SU(N)$. For the $SU(2)$ the λ^a are given by the Pauli matrices σ^i $(i = 1, 2, 3)$ and θ^a is a 3-component vector. For an element of the group $SU(2)$ we hence have

$$U = \exp\left(i\vec{\omega}\frac{\vec{\sigma}}{2}\right) \; . \tag{3.34}$$

For a general U we have

$$U^\dagger = \exp\left(-i\theta^a \left(\frac{\lambda^a}{2}\right)^\dagger\right) \stackrel{!}{=} U^{-1} = \exp\left(-i\theta^a \frac{\lambda^a}{2}\right) \; . \tag{3.35}$$

The generators must hence be Hermitean, *i.e.*

$$(\lambda^a)^\dagger = \lambda^a \; . \tag{3.36}$$

Furthermore, for the $SU(N)$ we have

$$\det(U) = 1 \; . \tag{3.37}$$

With

$$\det(\exp(A)) = \exp(\mathrm{Tr}(A)) \tag{3.38}$$

we have

$$\det\left(\exp\left(i\theta^a \frac{\lambda^a}{2}\right)\right) = \exp\left(i\theta^a \mathrm{Tr}\left(\frac{\lambda^a}{2}\right)\right) \stackrel{!}{=} 1 \; . \tag{3.39}$$

This implies

$$\mathrm{Tr}(\lambda^a) = 0 \; . \tag{3.40}$$

The generators of the $SU(N)$ must be traceless. The group $SU(N)$ has N^2-1 generators λ^a with $\mathrm{Tr}(\lambda^a)=0$. For the $SU(3)$ these are the Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} . \tag{3.41}$$

The matrices $\lambda^a/2$ are normalized as

$$\mathrm{Tr}\left(\frac{\lambda^a}{2}\frac{\lambda^b}{2}\right) = \frac{1}{2}\delta^{ab} . \tag{3.42}$$

For the Pauli matrices we have $(i=1,2,3)$

$$\mathrm{Tr}(\sigma_i^2) = 2 \qquad \text{and} \qquad \mathrm{Tr}(\sigma_1\sigma_2) = \mathrm{Tr}(i\sigma_3) = 0 . \tag{3.43}$$

Multiplied by $1/2$ they are the generators of the $SU(2)$. The generator matrices fulfill the completeness relation

$$\frac{\lambda^a_{ij}}{2}\frac{\lambda^a_{kl}}{2} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}\right) , \tag{3.44}$$

because

$$0 \stackrel{!}{=} \frac{\lambda^a_{ii}}{2}\frac{\lambda^a_{kl}}{2} = \frac{1}{2}\delta_{il}\delta_{ki} - \frac{1}{2N}\delta_{ii}\delta_{kl} = \frac{1}{2}\delta_{kl} - \frac{1}{2}\delta_{kl} = 0 . \tag{3.45}$$

3.3 Representations of Non-Abelian Gauge Groups

Be G a group with the elements $g_1, g_2 ... \in G$. An n-dimensional representation of G is given by the map $G \to C^{(n,n)}$, $g \to U(g)$. We have hence the map of abstract elements of the group onto complex $n \times n$ matrices so that $U(g_1 g_2) = U(g_1)U(g_2)$ holds and the properties of the group are preserved. A $U \in SU(N)$ can be written as $U = \exp(i\theta^a T^a)$. For the $SU(2)$ we hence have $U = \exp(i\vec{\omega}\cdot\vec{J})$. The group $SU(N)$ has N^2-1 generators T^a. For the $SU(2)$ these are the angular momentum operators J_i. The N^2-1 real parameters θ^a of the $SU(2)$ are given by $\vec{\omega}$. The fundamental representation of the $SU(2)$ is given by $J_i = \sigma_i/2$. In the general case we have $T^a = \lambda^a/2$. The generators fulfill the following commutation relation

$$[T^a, T^b] = i f^{abc} T^c . \tag{3.46}$$

The f^{abc} are the structure constants of the $SU(N)$ Lie algebra. They are totally antisymmetric and define $(N^2-1)(N^2-1)$ dimensional matrices $T^a_{lk} \equiv -i f^a_{lk} \equiv -i f^{alk}$. In case of the $SU(2)$ we have

$$[J_i, J_j] = \epsilon_{ijk} J_k . \tag{3.47}$$

Furthermore, we have

$$\mathrm{Tr}\left(\left[\frac{\lambda^a}{2},\frac{\lambda^b}{2}\right]\frac{\lambda^c}{2}\right) = if^{abe}\mathrm{Tr}\left(\frac{\lambda^e}{2}\frac{\lambda^c}{2}\right) = if^{abe}\frac{1}{2}\delta^{ec} = \frac{i}{2}f^{abc} \ . \tag{3.48}$$

The generators fulfill the Jacobi identity

$$[T^a,[T^b,T^c]] + [T^b,[T^c,T^a]] + [T^c,[T^a,T^b]] = 0 \ . \tag{3.49}$$

Applying Eq. (3.46) we obtain

$$0 = (-if^b_{cl})(-if^a_{lk}) + (-if^a_{lc})(-if^b_{lk}) + if^{abl}(-if^l_{ck}) \ . \tag{3.50}$$

And hence

$$0 = (T^bT^a)_{ck} - (T^aT^b)_{ck} + if^{abl}(T^l)_{ck} \ . \tag{3.51}$$

Thereby we have obtained an N^2-1 representation of the $SU(N)$ Lie algebra,

$$[T^a,T^b] = if^{abc}T^c \ . \tag{3.52}$$

This is the *adjoint representation*. We have the following $SU(N)$ representaions

- $d=1$: trivial representation (singlet).
- $d=N$: fundamental representation ($\lambda^a/2$), anti-fundamental representation ($-\lambda^{*a}/2$).
- $d=N^2-1$: adjoint representation.

3.4 Non-Abelian Gauge Transformations

Our starting point is the Lagrangian

$$\mathcal{L} = \sum_{i=1...N}\bar{\psi}_i(i\gamma^\mu\partial_\mu - m)\psi_i = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \qquad \text{with} \quad \bar{\Psi} = (\bar{\psi}_1,\bar{\psi}_2,...,\bar{\psi}_N) \ . \tag{3.53}$$

The Lagrangian is invariant under a global $SU(N)$ gauge transformation

$$\Psi \to \Psi' = \exp\left(i\theta^aT^a\right)\Psi = \left(1 + i\theta^aT^a + \mathcal{O}((\theta^a)^2)\right)\Psi = U\Psi \quad \text{and} \quad \bar{\Psi} \to \bar{\Psi}' = \bar{\Psi}U^{-1} \tag{3.54}$$

The generators T^a are

$$\begin{array}{lll} \text{fundamental representation:} & (T^a)_{ij} = \left(\frac{\lambda^a}{2}\right)_{ij} & d=N \\ \text{adjoint representation} & (T^a)_{bc} = -if^{abc} & d=N^2-1 \\ \text{trivial representation} & T^a = 0 \Leftrightarrow U(\theta)=1 \ . & \end{array} \tag{3.55}$$

We now consider local symmetries, hence $\theta^a = \theta^a(x)$. The transformation of Ψ be $\Psi' = U\Psi$. We introduce the covariant derivative

$$D_\mu = \partial_\mu - igA_\mu = \partial_\mu - igT^aA^a_\mu \ . \tag{3.56}$$

The T^a can change, but A^a_μ is identical in all D_μ. Example supersymmetry (SUSY):

$$\begin{array}{lll} \text{squark, quark} & T^a = \frac{\lambda^a}{2} & (d=N) \\ \text{gluino} & (T^a)_{bc} = -if^{abc} & (d=N^2-1) \end{array} \tag{3.57}$$

The covariant derivative transforms in the same way as Ψ, hence $(D_\mu\Psi)' = U(D_\mu\Psi)$. We have

$$(D_\mu\Psi)' = D'_\mu\Psi' = D'_\mu U\Psi \Rightarrow D'_\mu U = UD_\mu \tag{3.58}$$

This is fufilled if

$$\partial_\mu - igA'_\mu = D'_\mu = UD_\mu U^{-1} = U(\partial_\mu - igA_\mu)U^{-1} = UU^{-1}\partial_\mu + U(\partial_\mu U^{-1}) - igUA_\mu U^{-1} \Rightarrow \tag{3.59}$$

$$A'_\mu = \frac{i}{g}U(\partial_\mu U^{-1}) + UA_\mu U^{-1}\,, \tag{3.60}$$

with the A'^a_μ being independent of the representation of U. With infinitesimal

$$U = \exp(iT^a\theta^a) = 1 + iT^a\theta^a + \mathcal{O}(\theta^{a2}) \tag{3.61}$$

we have

$$\begin{aligned} A'_\mu &= A'^b_\mu T^b = \frac{i}{g}U(-i)T^a\,(\partial_\mu\theta^a)\,U^{-1} + \underbrace{(1+i\theta^a T^a)A^c_\mu T^c(1-i\theta^b T^b)}_{A^c_\mu T^c + iA^c_\mu \underbrace{(T^aT^c - T^cT^a)}_{if^{acb}T^b}\theta^a} \\ &= T^b\,\underbrace{(\frac{1}{g}\partial_\mu\theta^b + A^b_\mu + i(-if^{abc})\theta^a A^c_\mu)}_{A'^b_\mu}\,. \end{aligned} \tag{3.62}$$

The field strength tensor be defined through $F^{\mu\nu} \sim [D^\mu, D^\nu]$. We consider the commutator

$$\begin{aligned} [D_\mu, D_\nu] &= [\partial_\mu - igT^aA^a_\mu, \partial_\nu - igT^bA^b_\nu] = -igT^b\partial_\mu A^b_\nu - igT^a(-\partial_\nu A^a_\mu) + (-ig)^2 A^a_\mu A^b_\nu \underbrace{[T^a, T^b]}_{if^{abc}T^c} \\ &= -igT^a\underbrace{(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\underbrace{f^{bca}}_{f^{abc}}A^b_\mu A^c_\nu)}_{=:F^a_{\mu\nu}} = -igT^aF^a_{\mu\nu} = -igF_{\mu\nu}\,. \end{aligned} \tag{3.63}$$

The $F^a_{\mu\nu}$ are independent of the representation of T^a. We find for their transformation behaviour

$$F'_{\mu\nu} = \frac{i}{g}[D'^\mu, D'^\nu] = \frac{i}{g}[UD_\mu U^{-1}, UD_\nu U^{-1}] = UF_{\mu\nu}U^{-1} \quad \text{homogeneous transformation} \tag{3.64}$$

And with Eq. (3.62)

$$(F^a_{\mu\nu})' = F^a_{\mu\nu} + i(-if^{bac})\theta^b F^c_{\mu\nu} + ... \tag{3.65}$$

This means that $F^a_{\mu\nu}$ transforms homegeneously under the adjoint representation. Furthermore, it follows that

$$F^{a\mu\nu}F^a_{\mu\nu} = 2\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \left(= 2\mathrm{Tr}(F^{a\mu\nu}T^aF^b_{\mu\nu}T^b) = 2F^{a\mu\nu}F^b_{\mu\nu}\underbrace{\mathrm{Tr}(T^aT^b)}_{\frac{1}{2}\delta^{ab}} = F^{\mu\nu a}F^a_{\mu\nu} \right)$$

$$\text{is gauge invariant.} \tag{3.66}$$

With this we have for the kinetic Lagrangian

$$\mathcal{L}_{kin,A} = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} = -\frac{1}{2}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) \ . \tag{3.67}$$

Note that due to the non-linear term in $F^a_{\mu\nu}$ we have trilinear and quartic terms in the gauge fields, hence trilinear and quartic interactions among the gauge fields themselves. These additional interactions have important physical consequences.

3.5 Chiral Gauge Theories

We consider

$$\mathcal{L}_f = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi \ . \tag{3.68}$$

In the chiral representation the 4×4 γ matrices are given by

$$\gamma^\mu = \left(\begin{pmatrix} \mathbf{0} & \mathbb{1} \\ \mathbb{1} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} & -\vec{\sigma} \\ \vec{\sigma} & \mathbf{0} \end{pmatrix} \right) = \begin{pmatrix} 0 & \sigma^\mu_- \\ \sigma^\mu_+ & 0 \end{pmatrix} \tag{3.69}$$

$$\gamma^5 = \begin{pmatrix} \mathbb{1} & \mathbf{0} \\ \mathbf{0} & -\mathbb{1} \end{pmatrix} , \tag{3.70}$$

where σ_i $(i = 1, 2, 3)$ are the Pauli matrices. With

$$\Psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix} \quad \text{and} \quad \bar{\Psi} = \Psi^\dagger \gamma^0 = (\chi^\dagger, \varphi^\dagger) \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = (\varphi^\dagger, \chi^\dagger) \tag{3.71}$$

we have

$$\bar{\Psi} i\gamma^\mu D_\mu \Psi = i(\varphi^\dagger, \chi^\dagger) \underbrace{\begin{pmatrix} 0 & \sigma^\mu_- \\ \sigma^\mu_+ & 0 \end{pmatrix} \begin{pmatrix} D_\mu \chi \\ D_\mu \varphi \end{pmatrix}}_{\begin{pmatrix} \sigma^\mu_- D_\mu \varphi \\ \sigma^\mu_+ D_\mu \chi \end{pmatrix}} = \varphi^\dagger i \sigma^\mu_- D_\mu \varphi + \chi^\dagger i \sigma^\mu_+ D_\mu \chi \ . \tag{3.72}$$

The gauge interaction independently applies for

$$\Psi_L = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \frac{1}{2}(\mathbb{1} - \gamma_5)\Psi \quad \text{and} \quad \Psi_R = \begin{pmatrix} \chi \\ 0 \end{pmatrix} = \frac{1}{2}(\mathbb{1} + \gamma_5)\Psi \ . \tag{3.73}$$

The Ψ_L and Ψ_R can have independent gauge representations. But

$$m\bar{\Psi}\Psi = m(\varphi^\dagger, \chi^\dagger) \begin{pmatrix} \chi \\ \varphi \end{pmatrix} = m(\varphi^\dagger \chi + \chi^\dagger \varphi) = m(\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L) \ . \tag{3.74}$$

The mass term mixes Ψ_L and Ψ_R. This implies *symmetry breaking* if Ψ_L and Ψ_R have different representations.

What about the mass term for gauge bosons? We consider

$$\mathcal{L} = -\frac{1}{4} \underbrace{F^{a\mu\nu}F^a_{\mu\nu}}_{\text{gauge indep.}} + \frac{m^2}{2} \underbrace{A^{a\mu}A^a_\mu}_{\text{not gauge indep.}} \ . \tag{3.75}$$

For example for the $U(1)$

$$(A_\mu A^\mu)' = (A_\mu + \partial_\mu \theta)(A^\mu + \partial^\mu \theta) = A_\mu A^\mu + 2A_\mu \partial^\mu \theta + (\partial_\mu \theta)(\partial^\mu \theta) \ . \tag{3.76}$$

The mass term for A^μ breaks the gauge symmetry.

3.6 Example: The QCD Lagrangian

Quantum chromodynamics (QCD) is invariant under the colour $SU(3)$. The 6 quark fields carry colour charge and are in the fundamental representation

$$\Psi_q = \begin{pmatrix} \psi_{q1} \\ \psi_{q2} \\ \psi_{q3} \end{pmatrix} \qquad q = u, d, c, s, t, b \; . \tag{3.77}$$

They form triplets. The 8 gluons G^μ are in the adjoint representation The QCD Lagrangian is given by

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^{a\mu\nu} G^a_{\mu\nu} + \sum_{q=1...6} \bar{\Psi}_q (i\gamma^\mu D_\mu - m_q) \Psi_q \; , \tag{3.78}$$

with

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g f^{abc} G^b_\mu G^c_\nu \; . \tag{3.79}$$

Kapitel 4

Spontaneous Symmetry Breaking

We have seen in the previous discussion how gauge principles can serve as a dynamical principle to guide the construction of theories. Global gauge invariance implies via Noether's theorem the existence of a conserved current. Local gauge invariance requires the introduction of massless vector gauge bosons, fixes the form of the interactions of gauge bosons with sources and implies interactions among the gauge bosons in case of non-Abelian symmetries. We face the problem, however, that the gauge principle leads to theories in which all the interactions are mediated by massless vector bosons while only the photon and the gluons are massless and the vector bosons mediating the weak interactions, the W and Z bosons, are massive. We discuss in the following how this problem is solved by spontaneous symmetry breaking.

The symmetry of a Lagrangian is called *spontaneously broken* if the Lagrangian is symmetric but the physical vacuum *does not conserve* the symmetry. We will see that, if the Lagrangian of a theory is invariant under an exact continuous symmetry that is not the symmetry of the physical vacuum one or several massless spin-0 particles emerge. These are called *Goldstone bosons*. If the spontaneously broken symmetry is a local gauge symmetry the interplay (induced by the Higgs mechanism) between the *would-be Goldstone bosons* and the massless gauge bosons implies masses for the gauge bosons and removes the Goldstone bosons from the physical spectrum.

4.1 Example: Ferromagnetism

We consider a system of interacting spins,

$$H = -\sum_{i,j} J_{ij} \vec{S}_i \cdot \vec{S}_j \; . \tag{4.1}$$

The scalar product of the spin operators is a singlet with respect to rotations, *i.e.* rotation invariant. In the ground state of the ferromagnet (at sufficiently low temperature, below the Curie temperature) all spins are orientated along the same direction. This is the state with lowest energy. The ground state is no longer rotation invariant. Rotation of the system leads to a new ground state of same energy, which is different from the previous one, however. The ground state is degenerate. The distinction of a specific direction breaks the symmetry. We have spontaneous symmetry breaking here.

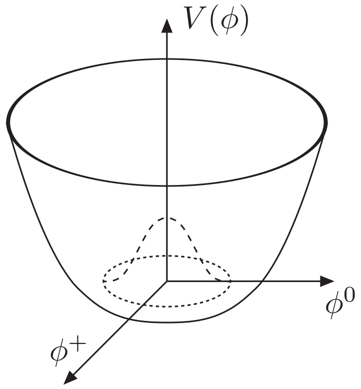

Abbildung 4.1: Das Higgspotential.

4.2 Example: Field Theory for a Complex Field

We consider the Lagrangian for a complex scalar field

$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2 \quad \text{with the potential} \quad V = \mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2 \,. \tag{4.2}$$

(Adding higher powers in ϕ leads to a non-renormalizable theory.) The Lagrangian is invariant under a $U(1)$ symmetry,

$$\phi \to \exp(i\alpha)\phi \,. \tag{4.3}$$

We consider the ground state. It is given by the minimum of V,

$$0 = \frac{\partial V}{\partial \phi^*} = \mu^2 \phi + 2\lambda(\phi^* \phi)\phi \quad \Rightarrow \quad \phi = \begin{cases} 0 & \text{für} \quad \mu^2 > 0 \\ \phi^* \phi = -\frac{\mu^2}{2\lambda} & \text{für} \quad \mu^2 < 0 \end{cases} \tag{4.4}$$

The parameter λ has to be positive so that the system does not become unstabel. For $\mu^2 < 0$ the potential takes the shape of a Mexican hat, see Fig. 4.1. At $\phi = 0$ we have a local maximum, at

$$|\phi| = v = \sqrt{-\frac{\mu^2}{2\lambda}} \tag{4.5}$$

a global minimum. Particles correspond to harmonic oscillations for the expansion about the minimum of the potential. Fluctuations into the direction of the (infinitely many degenerate) minima have the gradient zero and correspond to massless particles, the Goldstone bosons. Fluctuations perpendicular to this direction correspond to particles with mass $m > 0$. Expansion around the maximum at $\phi = 0$ would lead to particles with negative mass (tachyons), as the curvature of the potential is negative here.

Expansion about the minimum at $\phi = v$ leads to (we have two fluctuations φ_1 and φ_2 for the complex scalar field)

$$\phi = v + \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) = \left(v + \frac{1}{\sqrt{2}}\varphi_1\right) + i\frac{\varphi_2}{\sqrt{2}} \quad \Rightarrow \tag{4.6}$$

$$\phi^* \phi = v^2 + \sqrt{2} v \varphi_1 + \frac{1}{2}(\varphi_1^2 + \varphi_2^2) \,. \tag{4.7}$$

Thereby we obtain for the potential

$$V = \lambda(\phi^*\phi - v^2)^2 - \frac{\mu^4}{4\lambda} \qquad \text{with} \qquad v^2 = -\frac{\mu^2}{2\lambda} \quad \Rightarrow \tag{4.8}$$

$$V = \lambda\left(\sqrt{2}v\varphi_1 + \frac{1}{2}(\varphi_1^2 + \varphi_2^2)\right)^2 - \frac{\mu^4}{4\lambda} \, . \tag{4.9}$$

We neglect the last term in V as is is only a constant shift of the zero-point. We then obtain for the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi_1)^2 + \frac{1}{2}(\partial_\mu\varphi_2)^2 - 2\lambda v^2\varphi_1^2 - \sqrt{2}v\lambda\varphi_1(\varphi_1^2 + \varphi_2^2) - \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2 \, . \tag{4.10}$$

The terms quadratic in the fields provide the masses, the terms kubic and quartic in the fields are the interaction terms. We have a massive and a massless particle,

$$m_{\varphi_1} = 2v\sqrt{\lambda} \quad \text{and} \quad m_{\varphi_2} = 0 \, . \tag{4.11}$$

The massless particle is the Goldstone boson.

4.3 The Goldstone Theorem

The Goldstone theorem states:

In any field theory that obeys the 'usual axioms' including locality, Lorentz invariance and postive-definite norm on the Hilbert space, if an exact continuous symmetry of the Lagranian is not a symmetry of the physical vacuum, then the theory must contain a massless spin-zero particle (or particles) whose quantum numbers are those of the broken group generator (or generators).

Be

N = dimension of the algebra of the symmetry group of the complete Lagrangian.
M = dimension of the algebra of the group under which the vacuum is invariant after spontaneous symmetry breaking.

$\Rightarrow$ There are N-M Goldstone bosons without mass in the theory.

For each spontaneously broken degree of freedom of the symmetry there is a massless Goldstone boson.

4.4 Spontaneously broken Gauge Symmetries

We consider as example the Lagrangian of a complex sclar field Φ that couples to a photon field A_μ, that is invariant under a $U(1)$. The local transformations are given by

$$\Phi \to \exp(-ie\Lambda(x))\Phi(x) \qquad \text{and} \qquad A_\mu \to A_\mu + \partial_\mu\Lambda \, . \tag{4.12}$$

The Lagrangian reads

$$\mathcal{L} = [(\partial_\mu - ieA_\mu)\Phi^*][(\partial^\mu + ieA^\mu)\Phi] \underbrace{-\mu^2\Phi^*\Phi - \lambda(\Phi^*\Phi)^2}_{-V(\Phi)} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \, . \tag{4.13}$$

(Remark: In order to quantize the Lagrangian we additionally have to introduce a gauge fixing term.) For $\mu^2 < 0$ we have spontaneous symmmetry breaking of the $U(1)$. Then the field has a non-vanishing VEV,

$$< 0|\Phi|0 >= v = \sqrt{\frac{-\mu^2}{2\lambda}} \,. \tag{4.14}$$

The fluctuations around the minimum (expansion around the minimum) are given by

$$\Phi = v + \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) = \left(v + \frac{H(x)}{\sqrt{2}}\right) \exp\left(\frac{i}{\sqrt{2}}\frac{\chi(x)}{v}\right) \left(\approx v + \frac{1}{\sqrt{2}}(H(x) + i\chi(x))\right) \tag{4.15}$$

Thereby

$$\begin{aligned} D_\mu\Phi = (\partial_\mu + ieA_\mu)\Phi(x) &= \frac{1}{\sqrt{2}}(\partial_\mu\varphi_1 + i\partial_\mu\varphi_2) + ieA_\mu v + \frac{e}{\sqrt{2}}A_\mu(-\varphi_2 + i\varphi_1) \\ &= \exp\left(i\frac{\chi}{\sqrt{2}v}\right)\left[\partial_\mu + ie\left(A_\mu + \frac{\partial_\mu\chi}{\sqrt{2}ev}\right)\right]\left(v + \frac{H}{\sqrt{2}}\right) \,. \end{aligned} \tag{4.16}$$

In order to avoid bilinear mixing terms in the fields we perform the following gauge transformation

$$A'_\mu = A_\mu + \partial_\mu\left(\frac{\chi}{\sqrt{2}ev}\right) \,. \tag{4.17}$$

This results in the kinetic energy (from now on we call A' again A)

$$\begin{aligned} (D_\mu\Phi)^*(D^\mu\Phi) &= \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + e^2 A_\mu A^\mu\left(v + \frac{H}{\sqrt{2}}\right)^2 = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \underbrace{(e^2v^2)}_{\frac{1}{2}m_A^2} A_\mu A^\mu \\ &\quad \underbrace{+ e^2 A_\mu A^\mu\left(\sqrt{2}vH + \frac{H^2}{2}\right)}_{\text{interaction terms}} \,. \end{aligned} \tag{4.18}$$

And the complete Lagrangian reads

$$\begin{aligned} \mathcal{L} &= = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{1}{2}m_A^2 A_\mu A^\mu + e^2 A_\mu A^\mu\left(\sqrt{2}vH + \frac{H^2}{2}\right) \\ &\quad -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \underbrace{2\lambda v^2}_{\frac{1}{2}m_H^2} H^2 - \sqrt{2}v\lambda H^3 - \frac{\lambda}{4}H^4 \,. \end{aligned} \tag{4.19}$$

Here we have neglected the constant term λv^4 which simply shifts the zero-point of the vacuum. The masses of the Higgs particle H and the photon are

$$m_A^2 = 2e^2v^2 \tag{4.20}$$

$$m_H^2 = 4\lambda v^2 \,. \tag{4.21}$$

We hence have a massive photon (gauge boson) and a massive scalar field, the Higgs particle. The Goldstone boson does not appear any more as degree of freedom. The number of degrees of freedom has been preserved, however. Because in the unbroken $U(1)$ symmetry the photon is massless and has 2 physical degrees of freedom, the two transversal polarisations. The complex scalar field Φ has two degrees of freedom. When $U(1)$ is broken we have a massive photon with 3 degrees of freedom (including longitudinal polarisation) and a massive real Higgs particle with one degree of freedom. The Goldstone boson has been *eaten* to give mass to the photon, *i.e.* to provide the longitudinal degree of freedom of the massive gauge particle.

We summarise: In gauge theories Goldstone bosons do not appear. They are *would-be*) Goldstone bosons. Through spontaneous symmetry breaking they are directly absorbed into the longitudinal degrees of freedom of the massive gauge bosons. In gauge theories we have the following: Be

N = dimension of the algebra of the symmetry group of the complete Lagrangian.
M = dimension of the algebra of the group under which the vacuum is invariant after spontaneous symmetry breaking.
n = The number of the scalar fields.

$$\Rightarrow$$

There are M massless vector fields. (M is the dimension of the symmetry of the vacuum.)
There are $N-M$ massive vector fields. ($N-M$ is the number of broken generators.)
There are $n-(N-M)$ scalar Higgs fields.

The reason is that gauge theories do not satisfy the assumptions on which the Goldstone theorem is based. To quantize electrodynamics, for example, one must choose between the Gupta-Bleuler formalism with its unphysical indefinite metric states or quantization in a physical gauge wherein manifest covariance is lost.

4.5 Addendum: Goldstone Theorem - Classical Field Theory

Proof of the Goldstone theorem in classical field theory:
The Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - V(\varphi) \tag{4.22}$$

is invariant under the rotation

$$\varphi \to e^{-i\alpha_a R_a}\varphi \qquad a = 1, ..., N\,, \tag{4.23}$$

which can infinitesimally be written as

$$\varphi \to \varphi - i\alpha R\varphi \tag{4.24}$$

From the invariance it follows that

$$\delta V = \frac{\partial V}{\partial\varphi}\delta\varphi = -i\alpha\frac{\partial V}{\partial\varphi}R\varphi = 0 \qquad \forall\alpha, \varphi \tag{4.25}$$

so that

$$\frac{\partial^2 V}{\partial\varphi\partial\varphi}R\varphi + \frac{\partial V}{\partial\varphi}R = 0 \tag{4.26}$$

After spontaneous symmetry breaking we have the ground state

$$\frac{\partial V}{\partial \varphi} = 0 \qquad \text{for} \quad \varphi = v \neq 0 \tag{4.27}$$

from which follows the Goldstone equation:

$$\boxed{\frac{\partial^2 V}{\partial \varphi \partial \varphi} = 0 \qquad \text{for} \quad \varphi = v \qquad (4.28)}$$

and

$$\frac{\partial^2 V}{\partial \varphi \partial \varphi} \equiv M^2 \tag{4.29}$$

is the mass matrix of the system. Expanding φ about the ground state

$$\varphi = v + \varphi' \tag{4.30}$$

we have

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial\varphi)^2 - [V(v) + \overbrace{\frac{\partial V}{\partial \varphi}}^{0} \varphi' + \frac{1}{2}\varphi' \frac{\partial^2 V}{\partial \varphi \partial \varphi}\varphi' + ...] \\ &= \frac{1}{2}(\partial\varphi')^2 - \frac{1}{2}\varphi' \frac{\partial^2 V}{\partial \varphi \partial \varphi}\varphi' + ... \end{aligned} \tag{4.31}$$

The Goldstone equation is thus the condition equation for the masses

$$\underline{\underline{M^2 R v = 0}} \tag{4.32}$$

- The equation is fulfilled if the generators R^a, $a = 1, 2, ..., M$ leave the vacuum invariant: $R^a v = 0$.
- The remaining generators R^a, $a = M + 1, ..., N$ form a set of linearly independent vectors $R^a v$. These are eigen-vectors of the zero-eigenvalues of the mass matrix M^2. The zero-eigenvalue is hence $N - M$ times degenerated. Q.e.d.

Kapitel 5

The Standard Model of Particle Physics

The Standard Model of particle physics describes the today known basic building blocks of matter and (except for gravity) its interactions. These are the electromagnetic and the weak (the electroweak) and the strong interaction.

Before going into details we give a short historical overview of the steps towards the development of the electroweak theory by Sheldon Glashow, Abdus Salam and Steven Weinberg (1967).

5.1 A Short History of the Standard Model of Particle Physics

- Weak interaction: β decay [*A. Becquerel 1896, Nobel Prize 1903*[1]]

> *Antoine Henri Becquerel (15.12.1852 - 25.8.1908) was a French physicist, Nobel Prize winner and discovered radioactivity.*
>
> In 1896, while investigating fluorescence in uranium salts, Becquerel discovered radioactivity accidentally. Investigating the work of Wilhelm Conrad Röntgen, Becquerel wrapped a fluorescent mineral, potassium uranyl sulfate, in photographic plates and black material in preparation for an experiment requiring bright sunlight. However, prior to actually performing the experiment, Becquerel found that the photographic plates were fully exposed. This discovery led Becquerel to investigate the sponaneous emission of nuclear radiation.
>
> *In1903 he shared the Nobel Prize with Marie and Pierre Curie "in recognition of the extraordinary services he has rendered by his discovery of spontaneous radioactivity".*

$N \to N' + e^-$ violates energy and angular momentum conservation

> Lise Meitner and Otto Hahn showed in 1911 that the energy of the emitted electrons is continuous. Since the released energy is constant, one had expected a discrete spectrum. In order to explain this obvious energy loss (and also the violation of angular momentum conservation) Wolfgang Pauli proposed in 1930 in his letter of Dec 4 to the

[1] shared with Marie and Pierre Curie

"Dear radioactive ladies and gentlemen" (Lise Meitner et al.) the participation of a neutral, extremely light elementary particle (no greater than 1% the mass of a proton) in the decay process, which he called "neutron". Enrico Fermi changed this name 1931 in "neutrino", as a diminuation form of the nearly at the same time discovered heavy neutron.

Lise Meitner (7. 11.1878 - 27.10.1968) was an Austrian physicist who investigated radioactivity and nuclear physics. Otto Hahn (8.3.1879 - 28.7.1968) was a German chemist and received in 1944 the Nobel Prize in chemistry. Wolfgang Ernst Pauli (25.4.1900 - 15.12.1958) was an Austrian physicist.

- The neutrino hypothesis: [*W. Pauli 1930, Nobel Prize 1945*]

$N \to P + e^- + \bar{\nu}_e$
Spin = 1/2, Mass ≈ 0

In 1956 Clyde Cowan and Frederick Reines suceeded in the first experimental proof of the neutrino in one of the first big nuclear reactors.

Clyde Lorrain Cowan Jr (6.121919 - 24.5.1974) discoverd together with Frederick Reines the neutrino. Frederick Reines (6.3.1918 - 26.8.1998) was an American physicist and won in 1995 the Nobel Prize of phycsics in the name of the two of them

- Proof of Neutrino:

$$N \to P + e^- + \bar{\nu}_e \qquad \bar{\nu}_e + P \to N + e^+$$

The neutrino could be verified experimentally 1956 by Clyde L. Cowan and Frederick Reines in the inverse β decay ($\bar{\nu}_e + p \to e^+ + n$) at a nuclear reactor, which causes a much higher neutrino flux as radioactive elements in the β decay. (Nobel prize to Reines alone 1995, since Cowan died 1974.)
The muon neutrino was discovered 1962 by Jack Steinberger, Melvin Schwartz and Leon Max Lederman with the first produced neutrino beam at an accelerator. All three physicists received 1988 the Nobel Prize for their basic experiments about neutrinos - weakly interaction elementary particles with vanishing or very small rest mass.
In 2000, the tau-neutrino was found in the DONUT-experiment.

- The Fermi Theory [*E. Fermi, Nobel Prize 1938*]

Enrico Fermi developed a theory of weak interactions in analogy to quantum electrodynamics (QED), where four fermions directly interact with each other:
$\mathcal{L}_{\text{eff}} = \frac{G_F}{\sqrt{2}} J_\mu J^\mu$
[For small momentum tranfers the reactions can be approximated by a point-like interaction.]

Enrico Fermi (29.9.1901 - 28.11.1954) was an Itaiian physicist He received the Nobel Prize for physics in 1938 for his work on induced radioactivity'.

The Fermi interaction consists of 4 fermions directly interacting with each other. For example a neutron (or down quark) can split into an electron, anti-neutrino and proton (or up quark). Tree-level Feynman diagrams describe this interaction remarkably well. However, no loop diagrams can be taken into account, since the Fermi interaction is *not renormalizable*. The solution consists in replacing the 4-fermion interaction by a more complete theory - with an exchange of a W or Z boson like in the electroweak theory. This is then renormalizable. Before the electroweak theory was constructed George Sudarshan and Robert Marshak, and independently also Richard Feynman and Murray Gell-Mann were able to determine the correct tensor structure (vector minus axialvector $V - A$) of the 4-Fermi interaction.

- Die Yukawa Hypothesis: [*H. Yukawa, Nobel Prize 1949 for 'his prediction of mesons based on the theory of nuclear forces'*]

The pointlike Fermi coupling is the limiting case of the exchange of a "heavy photon" $\rightarrow W$ boson.

$\frac{G_F}{\sqrt{2}}$ pointlike coupling $\approx \frac{g^2}{m_W^2+Q^2} \approx \frac{g^2}{m_W^2}$ with exchange of a W−boson

Hideki Yukawa (23.1.1907 - 8.9.1981) was a Japanese theoretical physicist and the first Japanese to win the Nobel Prize.

Hideki Yukawa established the hypothesis, that nuclear forces can be explained through the exchange of a new hypothetic particle between the nucleons, in the same manner as the electromagnetic force between two electrons can well be described by the exchange of photons. However, this particle exchanging the nuclear force should not be massless (as are the photons), but have a mass of 100 GeV. This value can be estimated from the range of the nuclear forces: the bigger the mass of the particle, the smaller the range of the interaction transmitted by the particle. A plausible argument for this connection is given by the energy-time uncertainty principle.

- Parity violation in the weak interaction [*T.D. Lee, C.N. Yang, Nobel Prize 1957, und C.-S. Wu*]

The $\tau - \theta$ puzzle: Initially there were known two different positively charged mesons with strangeness ($S \neq 0$). These were distinguished based on their decay processes:

$$\begin{aligned} \Theta^+ &\rightarrow \pi^+\pi^0 \qquad && P_{2\pi} = +1 \\ \tau^+ &\rightarrow \pi^+\pi^+\pi^- \qquad && P_{3\pi} = -1 \end{aligned}$$

The final states of these two reactions have different parity. Since at that time it was assumed that parity is conserved in all reactions, the τ and θ would have had to be two different particles. However, precision measurements of mass and life time showed no difference between both particles. They seemed to be identical. The solution of this this $\theta - \tau$ puzzle was the parity violation of the weak interaction. Since both mesons decay via weak interaction, this reaction need not conserve parity contrary to the initial assumption. Hence, both decays could stem from the same particle, which was then named K^+.

$\Theta^+ = \tau^+ = K^+ \Rightarrow \mathcal{P}$ violated. (π has negative parity.)

Tsung-Dao Lee (born November 24, 1926) is a Chinese American physicist. In 1957, Lee with C. N. Yang received the Nobel Prize in Physics for their work on the violation of parity law in weak interaction, which Chien-Shiung Wu experimentally verified. Lee and Yang were the first Chinese Nobel Prize winners. Mrs Chien-Shiung Wu (31. Mai 1912 in Liuho, Province Jiangsu, China ; - 16. Februar 1997 in New York, USA) was a Chinese-American physicist.*

$V-A$ theory: On says, that parity is maximally violated. This means that the axial coupling has the same strength as the vectorial coupling: $|c_V| = |c_A|$. Since, as was shown in the Goldhaber experiment, there are only left-handed neutrinos and right-handed antineutrinos, one has rather: $c_V = -c_A$. This is why one calls the theory "$V-A$ theory".

- $\mathcal{CP}$ violation [*Cronin, Fitch, Nobel Prize 1980*]

$$\begin{aligned} K_L^0 &\rightarrow 3\pi \quad \mathcal{CP} = - \\ K_S^0 &\rightarrow 2\pi \quad \mathcal{CP} = + \end{aligned}$$

Details: After the discovery of parity violation it was supposed widely that $\mathcal{CP}$ is conserved. Assuming $\mathcal{CP}$ symmetry, the physical Kaon states are given by the $\mathcal{CP}$ eigenstates. The strong eigenstates K^0, $\bar{K}^0$ are, however, no $\mathcal{CP}$ eigenstates, since these two particles are their respective antiparticle. Hence, $\mathcal{CP}$ eigenstates are linear combinations of theses states.

$$|K_1^0> = \frac{1}{\sqrt{2}}(|K^0> - |\bar{K}^0>) \quad \text{with} \quad \mathcal{CP}|K_1^0> = |K_1^0> \tag{5.1}$$

$$|K_2^0> = \frac{1}{\sqrt{2}}(|K^0> + |\bar{K}^0>) \quad \text{with} \quad \mathcal{CP}|K_2^0> = -|K_2^0> \tag{5.2}$$

Supposing $\mathcal{CP}$ symmetry these states can only decay under $\mathcal{CP}$ conservation. For the neutral Kaons this leads to two different decay channels for K_1 and K_2, with very different phase spaces and hence very different lifetimes:

$$K_1^0 \rightarrow 2\pi \quad \text{(quick, since big phase space)} \tag{5.3}$$

$$K_2^0 \rightarrow 3\pi \quad \text{(slow, since small phase space)} \tag{5.4}$$

In fact, one has found two different species of neutral Kaons, which are very different in their lifetimes. These were named K_L^0 (long-lived, average lifetime $(5.16 \pm 0.04) \cdot 10^{-8}$ s) and K_S^0 (short-lived, average lifetime $(8.953 \pm 0.006) \cdot 10^{-11}$ s). The average lifetime of the long-lived Kaon is about a factor 600 larger than the one of the short-lived Kaon.

$\mathcal{CP}$ violation: Due to the supposed $\mathcal{CP}$ symmetry it was natural to identify the K_1^0, K_2^0 with K_S^0, K_L^0. Hence, the K_L^0 would always decay in three and never in two pions. But in reality James Cronin and Val Fitch found out 1964, that the K_L^0 decays with a small probaility (about 10^{-3}) also in two pions. This leads to the fact, that the physical states are no pure $\mathcal{CP}$ eigenstates, but contain a small amount ϵ of the other $\mathcal{CP}$ eigenstate, respectively. One has without normalization:

$$|K_S^0> = (|K_1^0> + \epsilon|K_2^0>) \tag{5.5}$$

$$|K_L^0> = (|K_2^0> + \epsilon|K_1^0>) \tag{5.6}$$

This phenomenon has been checked very carefully in experiments and is called $\mathcal{CP}$ *violation through mixing*, since it is given by the mixing of the $\mathcal{CP}$ eigenstates to the physical eigenstate. Cronin and Fitch received 1980 the Nobel prize for their discovery. Since one can conclude this $\mathcal{CP}$ violation only indirectly through the observation of the decay, it is also called **indirect $\mathcal{CP}$ violation**. Also **direct $\mathcal{CP}$ violation**, hence a violation directly in the observed decay, has been observed. The direct $\mathcal{CP}$ violation is for Kaons another factor of 1000 smaller than the indirect one and was shown experimentally only three decades later at the turn to the 21th century.

Val Logsdon Fitch (10. March 1923 in Merriman, Nebraska), American physicist. Fitch received 1980 together with James Cronin the physics Nobel Prize for the discovery of violations of fundamental symmetry principles in the decay of James Watson Cronin (* 29. September 1931 in Chicago), US-American physicist.*

- Glashow-Salam-Weinberg Theory (GSW): [*S.L. Glashow, A. Salam, S. Weinberg, Nobel Prize 1979*]

Sheldon Lee Glashow (5. December 1932 in New York) is a US-American physicist and Nobel prize winner. He received 1979 together with Abdus Salam and Steven Weinberg the physics Nobel prize for their work on the theory of the unification of the weak and electromagnetic interaction between elementary particles, including among others the prediction of the Z boson and the weak neutral currents. Abdus Salam (* 29. Januar 1926 in Jhang, Pakistan; - 21. November 1996 in Oxford, England) was a Pakistanian physicist and Nobel prize winner. Steven Weinberg (* 3. Mai 1933 in New York City) is a US-American physicist and Nobel prize winner.*

The electromagnetic interaction is the unified theory of quantum electrodynamics and the weak interaction. Together with quantum chromodynamics it is a pillow of the Standard Modell of physics. This unification was initially described theoretically by S.L. Glashow, A. Salam and S. Weinberg 1967. Experimentally the theory was confirmed 1973 indirectly through the discovery of the NC and 1983 through the experimental proof of the W and Z bosons. A peculiarity is the parity violation through the electroweak interaction.

5.2 Unitarity: Path to Gauge Theories

Fermi theory: μ, β decays, charged current (CC) reactions at small energies.

$$\mathcal{L}_{\text{eff}} = \frac{G_F}{\sqrt{2}} j_\lambda^* j^\lambda \qquad \begin{aligned} j_\lambda &= \bar{e}\gamma_\lambda(1-\gamma_5)\nu_e + (\mu) + (q) \\ G_F &= 1.16 \cdot 10^{-5}/\text{GeV}^2 \end{aligned}$$

CC scattering at high energies:

$$\sigma(\bar{\nu}_e e^- \to \mu^- \bar{\nu}_\mu) = \frac{G_F^2 s}{\pi}$$

s-wave unitarity $\quad \sigma_{LL} < \frac{4\pi}{s}$

[Partial-wave unitarity constrains the modulus of an inelastic partial-wave amplitude to be $|\mathcal{M}| < 1$. Make a partial-wave expansion of the scattering amplitude. The constraint is equivalent to $\sigma < \pi/p_{c.m.}^2$ for inelastic s-wave scattering.]

Domain of validity/unitarity constraint: $\quad \sqrt{s} < (2\pi/G_F)^{\frac{1}{2}} \sim 700$ GeV

$\Rightarrow$ 4 steps are necessary to construct of the Fermi theory a consistent field theory with attenuation of the 4-point coupling.

Although Fermi's phenomenological interaction was inspired by the theory of electromagnetism, the analogy was not complete, and one may hope to obtain a more satisfactory theory by pushing the analogy further. An obvious device is to assume that the weak interaction, like quantum electrodynamics, is mediated by vector boson exchange. The weak intermediate boson must have the following three properties:

(i) It carries charge ± 1, because the familiar manifestations of the weak interactions (such as β-decay) are charge-changing.

(ii) It must be rather massive, to reproduce the short range of the weak force.

(iii) Its parity must be indefinite.

1.) Introduction of charged $W^{\pm}$ bosons [*Yukawa*]:

Interaction range $\sim m_W^{-1} \Rightarrow$

$E \to \infty : \sigma \sim \frac{G_F^2 m_W^2}{\pi} \to$ partial-wave unitarity is fulfilled; $G_F = g_W^2/m_W^2$.

2.) Introduction of a neutral vector boson W^3 [*Glashow*]:

The introduction of the intermediate boson softens the divergence of the s-wave amplitude for the above process, it gives rise, however, to new divergences in other processes:

Production of longitudinally polarized W's in $\nu\bar{\nu}$ collisions.

$\epsilon_\lambda^L = (\frac{k_0}{m_W}, 0, 0, \frac{E}{m_W}) \approx \frac{k_\lambda}{m_W}$
$\sigma(\nu\bar{\nu} \to W_L W_L) \sim \frac{g_W^4}{s}\left(\frac{\sqrt{s}}{m_W}\right)^4 \sim \frac{g_W^4 s}{m_W^4}$

$\leftarrow$ violates unitarity for $\sqrt{s} \gtrsim 1$ TeV.

Solution: Introduction of a neutral W^3, coupled to fermions and $W^{\pm}$:

Condition for the disappearance of the linear s singularity:
$I^a_{ik} I^b_{kj} - I^b_{ik} I^a_{kj} - i f_{abc} I^c_{ij} = 0$

$\boxed{[I^a, I^b] = i f_{abc} I^c}$ The fermion-boson couplings form a Lie algebra [associated to a non-abelian group].

$$\left.\begin{array}{llll} \text{Fermion-boson coupling} & \sim & g_W \times & \text{representation matrix} \\ \text{Boson-boson coupling} & \sim & g_W \times & \text{structure constants} \end{array}\right\} g_W \text{ universal.}$$

3.) 4-point coupling:

$W_L W_L \to W_L W_L$

Amplitude $\sim g_W^2 f^2 \frac{s^2}{m_W^4} + ...$ compensated by: $-g_W^2 f^2 \frac{s^2}{m_W^4}$:
4-boson vertex: $\sim g_W^2 f \times f$

4.) Higgs particle: [*Weinberg, Salam*]

The remaining linear s divergence is canceled by the exchange of a scalar particle with a coupling $\sim$ mass of the source.

Amplitude $\sim -(g_W m_W)^2 \frac{1}{s} \left(\frac{\sqrt{s}}{m_W}\right)^4 \sim -g_W^2 \frac{s}{m_W^2}$

The same mechanism cancels the remaining singularity in $f\bar{f} \to W_L W_L$ (f massive!)

Adding up the gauge diagrams we are left with $\sim g_W^2 \frac{m_f \sqrt{s}}{m_W^2}$

u	c	t	} Quarks
d	s	b	
ν_e	ν_μ	ν_τ	} Leptons
e	μ	τ	
1.	2.	3.	Family

Tabelle 5.1: Matter particles of the Standard Model.

$$\text{scalar diagram} \sim \sqrt{s}\left(g_W \frac{m_f}{m_W}\right)\frac{1}{s}(g_W m_W)\left(\frac{\sqrt{s}}{m_W}\right)^2 \sim g_W^2 \frac{\sqrt{s} m_f}{m_W^2}$$

Summary:
A theory of massive gauge bosons and fermions that are weakly coupled up to very high energies, requires, by unitarity, the existence of a Higgs particle; the Higgs particle is a scalar 0^+ *particle that couples to other particles proportionally to the masses of the particles.*

$\Rightarrow$ Non-abelian gauge field theory with spontaneous symmetry breaking.

5.3 Gauge Symmetry and Particle Content

The underlying gauge symmetry of the SM is the $SU(3)_C \times SU(2)_L \times U(1)_Y$. The $SU(3)_C$ describes QCD. The conserved charge associated with QCD is the colour charge. The corresponding gauge bosons that mediate the interaction (and are hence called interaction particles) are the 8 massless gluons. The $SU(2)_L$ describes the weak isospin interactions acting only between the left-handed fermions, and $U(1)_Y$ the weak hypercharge interactions that differ between the left- and right-handed fermions. The isospin and hypercharge interactions are partly unified in the electroweak interactions which are spontaneously broken. The electroweak interaction is mediated by three W boson fields and one B field associated with hypercharge. These fields mix to form two charged $W^\pm$ bosons and the neutral Z boson of the weak interactions and the photon γ of the electromagnetic interactions. These particles are interaction particles and carry spin 1. The conserved charges associated with the electroweak sector are the weak isospin and the weak hypercharge.

The particle content is given by the *matter* particles and the *interaction particles*. The matter particles are fermions with spin 1/2 and are subdivided in three families. They comprise 6 quarks and 6 leptons. We know three up-type (up, charm, top) and three down-type (down, strange, bottom) quarks. The leptons consist of three charged (e, μ, τ) and three neutral leptons, the neutrinos (ν_e, ν_μ, ν_τ), *cf.* Table 5.1.

The 3 lepton and quark families have identical quantum numbers, respectively, and are only distinguished through their masses. Therefore, when discussing the gauge interaction is is sufficient to consider only one family. The transformation behaviour of the quark and lepton fields under the SM gauge groups is summarized (for one generation) in Table 5.2.

The masses of the particles are generated through spontaneous symmetry breaking (SSB). For this a complex Higgs doublet ($d_D = 4$ degrees of freedom) is added together with the

Field	$U(1)_Y \times SU(2)_L \times SU(3)_C$
$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$(\frac{1}{3}, \mathbf{2}, \mathbf{3})$
u_R	$(\frac{2}{3}, \mathbf{1}, \bar{\mathbf{3}})$
d_R	$(-\frac{4}{3}, \mathbf{1}, \bar{\mathbf{3}})$
$L_L = \begin{pmatrix} e_L \\ \nu_{e_L} \end{pmatrix}$	$(-1, \mathbf{2}, \mathbf{1})$
e_R	$(2, \mathbf{1}, \bar{\mathbf{1}})$

Tabelle 5.2: Transformation behaviour under the SM gauge groups.

Higgs potential V. The SSB breaks down the $SU(2)_L \times U(1)_Y$ ($d_{EW} = 4$) to the electromagnetic $U(1)_{em}$ ($d_{em} = 1$). The electromagnetic charge hence remains conserved. Associated with this SSB are $d_{EW} - d_{em} = 4 - 1 = 3$ would-be Goldstone bosons that are absorbed to give masses to the $W^\pm$ and Z bosons. The photon remains massless. Furthermore, after SSB there are $d_D - (d_{EW} - d_{em}) = 4 - (4 - 1) = 4 - 3 = 1$ Higgs particles in the spectrum.

One last remark is at order: We know that the neutrinos have mass. When we formulate the SM in the following we will neglect the neutrino mass and assume neutrinos to be massless. For the treatment of massive neutrinos we refer to the literature.

5.4 Glashow-Salam-Weinberg Theory for Leptons

We only consider the first lepton generation, *i.e.* e, ν_e. The generalization to the other generations is trivial. We have the

electromagnetic interaction:

$$\mathcal{L}_{int} = -e_0 j_\mu^{elm} A^\mu \quad \text{with} \tag{5.7}$$

$$j_\mu^{elm} = -\bar{e}\gamma_\mu e \, , \tag{5.8}$$

where e_0 denotes the elementary charge with $\alpha = e_0^2/4\pi$. And we have the

weak interaction:

$$\mathcal{L}_W = -\frac{4G_F}{\sqrt{2}} j_\mu^- j^{\mu+} \tag{5.9}$$

in the Fermi notation for charged currents,

with

$$j_\mu^+ = \bar{\nu}_e \gamma_\mu \frac{1-\gamma_5}{2} e = \bar{\nu}_{eL}\gamma_\mu e_L \qquad \text{(left-chiral)} \tag{5.10}$$

$$j_\mu^- = (j_\mu^+)^* \tag{5.11}$$

G_F denotes the Fermi constant, $G_F \approx 10^{-5}/m_P^2$.

The next steps are:

- Resolve the 4-Fermi coupling through the exchange of a heavy vector boson. Apart from the vector boson mass the structure of the weak interaction is similar to the one of electrodynamics.
- Construction of the theory as gauge field theory with spontaneous symmetry breaking, to guarantee renormalizability.
- Analysis of the physical consequences of the symmetry and its breaking.

The free Lagrangian for the electrons and left-handed neutrinos[2] is given by the following expression that takes into account that the particles are massless in case of chiral invariance,

$$\begin{aligned}\mathcal{L}_0 &= \bar{e} i\partial\!\!\!/ e + \bar{\nu}_{eL} i\partial\!\!\!/ \nu_{eL} \\ &= \bar{e}_L i\partial\!\!\!/ e_L + \bar{e}_R i\partial\!\!\!/ e_R + \bar{\nu}_{eL} i\partial\!\!\!/ \nu_{eL}\ ,\end{aligned} \tag{5.12}$$

where

$$f_{R,L} = \frac{1}{2}(1 \pm \gamma_5) f \tag{5.13}$$

The free Lagrangian $\mathcal{L}_0$ is $SU(2)_L$ symmetric. The associated conserved charge is the weak isospin:

$$\begin{aligned}\begin{pmatrix}\nu_e \\ e\end{pmatrix}_L &: \text{ Isodublett mit } \quad I(\nu_{eL}) = I(e_L) = \frac{1}{2} \quad \text{and} \quad I_3(\nu_{eL}) = +\frac{1}{2} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad I_3(e_L) = -\frac{1}{2} \\ e_R &: \text{ Isosingulett mit } \quad I(e_R) = I_3(e_R) = 0\end{aligned} \tag{5.14}$$

The Lagrangian

$$\mathcal{L}_0 = \overline{\begin{pmatrix}\nu_e \\ e\end{pmatrix}}_L i\partial\!\!\!/ \begin{pmatrix}\nu_e \\ e\end{pmatrix}_L + \bar{e}_R i\partial\!\!\!/ e_R \tag{5.15}$$

is invariant under the global isospin transformation

$$\begin{aligned}\begin{pmatrix}\nu_e \\ e\end{pmatrix}_L &\rightarrow e^{-\frac{i}{2} g\vec{\alpha}\vec{\tau}} \begin{pmatrix}\nu_e \\ e\end{pmatrix}_L \\ e_R &\rightarrow e_R\end{aligned} \tag{5.16}$$

The theory becomes locally $SU(2)_L$ invariant through the introduction of an isovector $\vec{W}_\mu$ of vector fields with minimal coupling:

$$\left.\begin{aligned}\text{doublet} &: \ i\partial\!\!\!/ \rightarrow i\partial\!\!\!/ - \tfrac{g}{2}\vec{\tau}\vec{W\!\!\!/} \\ \text{singlet} &: \ i\partial\!\!\!/ \rightarrow i\partial\!\!\!/\end{aligned}\right\} \qquad \text{from:} \quad i\partial\!\!\!/ \rightarrow i\partial\!\!\!/ - g\vec{I}\vec{W\!\!\!/} \tag{5.17}$$

[2]The Goldhaber experiment (1957) has shown, that neutrinos appear in nature only as left-handed particles. This is a confirmation of the $V-A$ theory that predicts the parity violation of the weak interaction.

The resulting interaction Lagrangian for the lepton-W coupling reads:

$$\begin{aligned}
\mathcal{L}_{int} &= -\frac{g}{2}\overline{\begin{pmatrix}\nu_e\\ e\end{pmatrix}}_L \gamma_\mu \vec{\tau} \begin{pmatrix}\nu_e\\ e\end{pmatrix}_L \vec{W}^\mu \\
&= -\frac{g}{2\sqrt{2}}\bar{\nu}_e\gamma_\mu(1-\gamma_5)eW^{+\mu} + h.c. - \frac{g}{4}\{\bar{\nu}_e\gamma_\mu(1-\gamma_5)\nu_e - \bar{e}\gamma_\mu(1-\gamma_5)e\}W^{3\mu} \qquad (5.18)
\end{aligned}$$

where we have introduced

$$W^\pm = \frac{1}{\sqrt{2}}(W^1 \mp iW^2)\ . \qquad (5.19)$$

From Eq. (5.18) we can read off

- The charged lepton current has per construction the correct structure.
- W^3_μ, the neutral isovector field cannot be identified with the photon field A_μ since the electromagnetic currect does not contain any ν's and furthermore has a pure vector character (and hence does not contain a γ_5).

This leads to the formulation of the minimal $SU(2)_L \times U(1)_Y$ gauge theory:

The Lagrangian $\mathcal{L}_0$, Eq. (5.18), has an additional $U(1)$ gauge symmetry (after coupling $\vec{W}$) and associated with this the weak hypercharge. The quanum numbers are defined in such a way that we obtain the correct electromagnetic current:
(In order to include electromagnetism we define the "weak hypercharge".)

$$\begin{aligned}
j^{elm}_\mu = -\bar{e}\gamma_\mu e &= -\bar{e}_L\gamma_\mu e_L - \bar{e}_R\gamma_\mu e_R \\
&= \underbrace{\frac{1}{2}\overline{\begin{pmatrix}\nu_e\\ e\end{pmatrix}}_L \gamma_\mu\tau_3 \begin{pmatrix}\nu_e\\ e\end{pmatrix}_L}_{\substack{\text{Isovector current,}\\ \text{coupling to } W^3_\mu}} \underbrace{- \frac{1}{2}\overline{\begin{pmatrix}\nu_e\\ e\end{pmatrix}}_L \gamma_\mu 1 \begin{pmatrix}\nu_e\\ e\end{pmatrix}_L - \bar{e}_R\gamma_\mu e_R}_{\substack{\text{Isosinglets, for the construction}\\ \text{of the hypercharge current}}} \qquad (5.20)
\end{aligned}$$

The hypercharge quantum numbers are

$$Y(\nu_{eL}) = Y(e_L) = -1 \qquad (5.21)$$

$$Y(e_R) = -2\ . \qquad (5.22)$$

This follows from the requirement that the Gell-Mann Nishijima relation[3] holds

$$Q = I_3 + \frac{1}{2}Y \qquad (5.23)$$

Local gauge invariance is achieve through the minimal coupling of the gauge vector field,

$$i\partial\!\!\!/ \to i\ \partial\!\!\!/ - \frac{g'}{2}Y B\!\!\!/\ . \qquad (5.24)$$

[3] Originally this equation was derived from empiric observations. Nowadays it is understood as result of the quark model.

This leads to the Lagrangian

$$\mathcal{L}_{int} = -\frac{g}{\sqrt{2}}\bar{\nu}_{eL}\gamma_\mu e_L W^{+\mu} + h.c. \quad - \quad \frac{g}{2}\{\bar{\nu}_{eL}\gamma_\mu \nu_{eL} - \bar{e}_L\gamma_\mu e_L\}W^{3\mu}$$
$$+ \quad g'\{\frac{1}{2}\bar{\nu}_{eL}\gamma_\mu \nu_{eL} + \frac{1}{2}\bar{e}_L\gamma_\mu e_L + \bar{e}_R\gamma_\mu e_R\}B^\mu \qquad (5.25)$$

From the Lagrangian Eq (5.25) we can read off:

- The charged currents remain unchanged.
- We can introduce a mixture between W^3 and B in such a way that the pure parity invariant electron photon interaction is generated. We are left with a neutral current interaction with the orthogonal field combination:

$$\left.\begin{array}{rcl} A_\mu &=& \cos\theta_W B_\mu + \sin\theta_W W^3_\mu \\ Z_\mu &=& -\sin\theta_W B_\mu + \cos\theta_W W^3_\mu \end{array}\right\} \qquad \left.\begin{array}{rcl} B_\mu &=& \cos\theta_W A_\mu - \sin\theta_W Z_\mu \\ W^3_\mu &=& \sin\theta_W A_\mu + \cos\theta_W Z_\mu \end{array}\right\} \qquad (5.26)$$

Here θ_W denotes the Weinberg angle. Rewriting the Lagrangian in terms of A_μ and Z_μ leads to the A_μ coupling

$$A_\mu\{\bar{\nu}_{eL}\gamma_\mu\nu_{eL}\{-\frac{g}{2}\sin\theta_W + \frac{g'}{2}\cos\theta_W\} + \bar{e}_L\gamma_\mu e_L\{\frac{g}{2}\sin\theta_W + \frac{g'}{2}\cos\theta_W\} + \bar{e}_R\gamma_\mu e_R g'\cos\theta_W\} \qquad (5.27)$$

The neutrino ν can be eliminated through

$$\tan\theta_W = \frac{g'}{g}\,. \qquad (5.28)$$

(The photon only couples to charge particles!) The correct e-coupling is obtained by

$$\left.\begin{array}{rcl} g'\cos\theta_W &=& e_0 \\ g\sin\theta_W &=& e_0 \end{array}\right\} \frac{1}{e_0^2} = \frac{1}{g^2} + \frac{1}{g'^2} \qquad (5.29)$$

The lepton-boson interaction hence reads

$$\begin{aligned} \mathcal{L}_{int} &= -\frac{g}{2\sqrt{2}}\bar{\nu}_e\gamma_\mu(1-\gamma_5)eW^{+\mu} + h.c. \\ &- \frac{g}{4\cos\theta_W}\{\bar{\nu}_e\gamma_\mu(1-\gamma_5)\nu_e - \bar{e}\gamma_\mu(1-\gamma_5)e + 4\sin^2\theta_W\bar{e}\gamma_\mu e\}Z^\mu \qquad (5.30) \\ &+ e_0\bar{e}\gamma_\mu e A^\mu \end{aligned}$$

The first line describes the charged current interactions, the second the neutral current interaction and the third line the electromagnetic ineractions.

The coupling constants of the theory are: $[g, g']$ or $[e_0, \sin\theta_W]$.

- The coupling $e_0 = \sqrt{4\pi\alpha} \sim \frac{1}{3}$ is fixed within electromagnetism.
- The second parameter is not fixed through the weak interactions as the charged current only fixes the relation $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}$.

With the notation

$$\begin{aligned} j_\mu^- &= \bar{\nu}_e \gamma_\mu \frac{1-\gamma_5}{2} e \\ j_\mu^3 &= \overline{\begin{pmatrix} \nu_e \\ e \end{pmatrix}}_L \gamma_\mu \frac{\tau^3}{2} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \\ j_\mu^{em} &= -\bar{e}\gamma_\mu e \end{aligned} \tag{5.31}$$

the interaction Lagrangian can be written as

$$\boxed{\begin{aligned} \mathcal{L}_{int} &= -\frac{g}{\sqrt{2}} j_\mu^- W^{+\mu} + h.c. \\ &\quad -\frac{g}{\cos\theta_W}\{j_\mu^3 - \sin^2\theta_W j_\mu^{em}\} Z^\mu \qquad (5.32) \\ &\quad -e_0 j_\mu^{em} A^\mu \end{aligned}}$$

where $\underline{g = \frac{e_0}{\sin\theta_W}}$.

However, the Lagrangian does not conain mass terms for the fermions and gauge bosons yet. The theory muss be modified in such a way that the particles obtain their mass without getting into conflict with the gauge symmetry underlying the theory.

5.5 Introduction of the W, Z Boson and Fermion Masses

Let us repeat. With the currents

$$\begin{aligned} j_\mu^\pm &= \bar{l}_L \gamma_\mu \tau^\pm l_L \qquad \text{where} \quad l_L = (\nu_e, e)_L^T \\ j_\mu^3 &= \bar{l}_L \gamma_\mu \frac{1}{2} \tau^3 l_L \qquad (5.33) \\ j_\mu^{em} &= -\bar{e}\gamma_\mu e \qquad (5.34) \end{aligned}$$

the interaction Lagrangian can be written as

$$\begin{aligned} \mathcal{L}_{int} &= -\frac{g}{\sqrt{2}} j_\mu^- W^{+\mu} + h.c. \\ &\quad -\frac{g}{\cos\theta_W}\{j_\mu^3 - \sin^2\theta_W j_\mu^{em}\} Z^\mu \qquad (5.35) \\ &\quad -e_0 j_\mu^{em} A^\mu \qquad (5.36) \end{aligned}$$

and the couplings fulfill the relations

$$\begin{aligned} \frac{g'}{g} &= \tan\theta_W \\ \frac{G_F}{\sqrt{2}} &= \frac{g^2}{8m_W^2} \\ e_0 &= g\sin\theta_W \,. \end{aligned} \tag{5.37}$$

The generation of masses for the 3 vector fields, hence the absorption of 3 Goldstone bosons, is not possible with 3 scalar fields. The minimal solution is the introcduction of one complex doublet with 4 degrees of freedom,

$$\phi = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} \qquad \text{with} \qquad \begin{array}{rcl} \phi_+ & = & \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\ \phi_0 & = & \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4) \end{array} \tag{5.38}$$

The Lagrangian of the doublet field ϕ is given by

$$\underline{\underline{\mathcal{L}_\phi = \partial_\mu \phi^* \partial^\mu \phi - \mu^2 \phi^* \phi - \lambda(\phi^*\phi)^2}} \tag{5.39}$$

It is $SU(2)_L \times U(1)_Y$ invariant. The field ϕ transforms as

$$\phi \to e^{-\frac{i}{2} g \vec{\alpha}\vec{\tau}} e^{-\frac{i}{2} g' \beta} \, .\phi \tag{5.40}$$

After spontaneous symmetry breaking the vacuum expectation value of the scalar field is

$$< \phi > = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \qquad v^* = v \tag{5.41}$$

It breaks the $SU(2)_L \times U(1)_Y$ symmetry, but is invariant under the $U(1)_{em}$ symmetry, generated by the electric charge operator. Since each (would-be) Goldstone boson is associated with a generator that breaks the vacuum, we have $4 - 1 = 3$ Goldstone bosons. The quantum numbers of the field ϕ are

$$\left.\begin{array}{rclr} I_3(\phi_+) & = & +\frac{1}{2} & Y(\phi_+) = +1 \\ I_3(\phi_0) & = & -\frac{1}{2} & Y(\phi_0) = +1 \end{array}\right\} \begin{array}{rcl} Q(\phi_+) & = & 1 \\ Q(\phi_0) & = & 0 \end{array} \tag{5.42}$$

(The field ϕ transforms as an $SU(2)_L$ doublet and therefore has to have the hypercharge $Y_\phi = 1$.) The gauge fields are introduced through minimal coupling,

$$i\partial_\mu \to i\partial_\mu - \frac{g}{2}\vec{\tau}\vec{W}_\mu - \frac{g'}{2} B_\mu \, . \tag{5.43}$$

Expanding about the minimum of the Higgs potential

$$\begin{array}{c} \phi_+(x) \to 0 \\ \phi_0(x) \to \frac{1}{\sqrt{2}}[v + \chi(x)] \qquad \chi* = \chi \end{array} \tag{5.44}$$

one obtains from the kinetic part of the Lagrangian of the scalar field

$$\begin{aligned} \mathcal{L}_m &= \left| \left[(i\frac{g}{2}\vec{\tau}\vec{W} + i\frac{g'}{2}B) \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \right] \right|^2 \\ &= \frac{1}{2}\frac{v^2}{4} \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ B \end{pmatrix}^T \begin{pmatrix} g^2 & & & \\ & g^2 & & \\ & & g^2 & -gg' \\ & & -gg' & g'^2 \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ B \end{pmatrix} \end{aligned} \tag{5.45}$$

with the eigenvalues of the mass matrix given by

$$\begin{aligned} m_1^2 &= m_2^2 = \frac{g^2 v^2}{4} \\ m_3^2 &= \frac{(g^2 + g'^2)v^2}{4} \\ m_4^2 &= 0 \end{aligned} \tag{5.46}$$

Thereby the masses of the gauge bosons read

$$
\boxed{
\begin{aligned}
m_\gamma^2 &= 0 && (5.47)\\
m_W^2 &= \frac{1}{4} g^2 v^2 && (5.48)\\
m_Z^2 &= \frac{1}{4}(g^2 + g'^2) v^2 && (5.49)
\end{aligned}
}
$$

They fulfill the folowing <u>mass relations</u>:

(i) W boson mass: We have $e_0^2 = g^2 \sin^2\theta_W = 4\sqrt{2} G_F \sin^2\theta_W m_W^2$, so that

$$
\boxed{m_W^2 = \frac{\pi\alpha}{\sqrt{2}G_F}\frac{1}{\sin^2\theta_W}} \qquad (5.50)
$$

with $\alpha \approx \alpha(m_Z^2)$ (effective radiative correction). With $\sin^2\theta_W \approx 1/4$ the W boson mass is $m_W \approx 80$ GeV.

(ii) Z boson mass: With

$$
\frac{m_W^2}{m_Z^2} = \cos^2\theta_W \qquad (5.51)
$$

we obtain

$$
\boxed{\sin^2\theta_W = 1 - \frac{m_W^2}{m_Z^2}} \qquad (5.52)
$$

Finally one obtains with Eq. (5.48) for the <u>Higgs vacuum expectation value</u>

$$
\frac{1}{v^2} = \frac{g^2}{4m_W^2} = \sqrt{2}G_F \qquad (5.53)
$$

and thereby

$$
\boxed{v = \frac{1}{\sqrt{\sqrt{2}G_F}} \approx 246 \text{ GeV}} \qquad (5.54)
$$

The vacuum expectation value v is the characteristic scale of electroweak symmetry breaking.

<u>The Higgs mechanism for charged lepton masses:</u> The fermions couple via the gauge-invariant Yukawa coupling to the Higgs field ϕ:

The interaction Lagrangian reads

$$
\mathcal{L}(ee\phi) = -f_e \overline{\begin{pmatrix} \nu_e \\ e \end{pmatrix}}_L \phi e_R + h.c. \qquad (5.55)
$$

It is invariant under $SU(2)_L \times U(1)_Y$. After expansion of the Higgs field around the VEV one obtains

$$\begin{aligned}\mathcal{L}(ee\Phi) &= -f_e\frac{v}{\sqrt{2}}[\bar{e}_L e_R + \bar{e}_R e_L] + ... \\ &= -f_e\frac{v}{\sqrt{2}}\bar{e}e + ... \\ &= -m_e\bar{e}e + ... \end{aligned} \tag{5.54}$$

The electron mass is given by

$$\boxed{m_e = \frac{f_e v}{\sqrt{2}} \qquad (5.55)}$$

5.6 Quarks in the Glashow-Salam-Weinberg Theory

In this chapter the hadronic sector is implemented in the SM of the weak and electromagnetic interactions. This is done in the context of the quark model. Since quarks and leptons ressemble each other, the construction on the quark level is obvious, but not trivial.

We know from the previous chapters that the lepton currents are built from multiplets.

$$\boxed{\begin{pmatrix}\nu_e \\ e^-\end{pmatrix}_L \quad e_R^- \quad \begin{pmatrix}\nu_\mu \\ \mu^-\end{pmatrix}_L \quad \mu_R^- \quad \begin{pmatrix}\nu_\tau \\ \tau^-\end{pmatrix}_L \quad \tau_R^- \qquad (5.56)}$$

This can be generalized to the quark currents.

For the quark currents for u, d, s we have:

1) The electromagnetic current, after summation over all possible charges, is given by

$$j_\mu^{elm} = \sum_{Q_q} Q_q \bar{q}\gamma_\mu q = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s \tag{5.57}$$

2) From low-energy experiments (pion and Kaon decays) it followed that the left-handed weak current, the Cabibbo current, is given by[4]

$$\begin{aligned} j_\mu^- &= \cos\theta_c \bar{u}\gamma_\mu \frac{1}{2}(1-\gamma_5)d + \sin\theta_c \bar{u}\gamma_\mu \frac{1}{2}(1-\gamma_5)s \\ &= \bar{u}\gamma_\mu \frac{1}{2}(1-\gamma_5)[\cos\theta_c d + \sin\theta_c s] \end{aligned} \tag{5.57}$$

[4]Cabibbo's conjecture was that the quarks that participate in the weak interactions are a mixture of the quarks that participate in the strong interaction. The mixing was originally postulated by Cabibbo (1963) to explain certain decay patterns in the weak interactions and originally had only to do with the d and s quarks.

with $\sin^2\theta_c \approx 0.05$. We define the Cabibbo rotated quarks

$$\begin{aligned} d_c &= \cos\theta_c d + \sin\theta_c s \\ s_c &= -\sin\theta_c d + \cos\theta_c s \end{aligned} \tag{5.57}$$

Here,

d, s are different direction in the (u, d, s) space of quarks, characterized by different masses, *i.e.* we are in the mass basis.

d_c, s_c are directions in the quark space, characterized through the weak interacation, they represent the current basis.

The current $j_\mu^\pm$ can be expressed through $j_\mu^\mp = \bar{Q}_L\gamma_\mu\tau^\mp Q_L$ with the definitions of the multiplets given by

$$\begin{pmatrix} u \\ d_c \end{pmatrix}_L \quad s_{cL} \quad \begin{matrix} u_R \\ d_{cR} \end{matrix} \quad s_{cR} \tag{5.58}$$

3) The corresponding neutral isovector current is then given by

$$\begin{aligned} j_\mu^3 &= \sum_{doublets} \bar{Q}_L\gamma_\mu\frac{1}{2}\tau^3 Q_L \\ &\sim \bar{u}_L\gamma_\mu u_L - \bar{d}_{cL}\gamma_\mu d_{cL} \\ &= \bar{u}_L\gamma_\mu u_L - \cos^2\theta_c\bar{d}_L\gamma_\mu d_L - \sin^2\theta_c\bar{s}_L\gamma_\mu s_L \\ &\quad -\sin\theta_c\cos\theta_c[\bar{d}_L\gamma_\mu s_L + \bar{s}_L\gamma_\mu d_L] \end{aligned} \tag{5.56}$$

The first line is a diagonal neutral current. The second line is a strangeness changing neutral current with the strength $\sim \sin\theta_c$, like the strangeness changing charged current.

This is in striking contradiction with the experimental non-observation of strangeness changing neutral current reactions. There are strict experimental limits on the decay rates that are mediated by strangeness changing neutral currents like

$$1)\ \frac{\Gamma(K_L \to \mu^+\mu^-)}{\Gamma(K^+ \to \mu^+\nu_\mu)} < \sim 4\cdot 10^{-9}(\text{exp})$$

$$2)\ \frac{\Gamma(K^+ \to \pi\nu\bar{\nu})}{\Gamma(K^+ \to all)} < 1.4\cdot 10^{-7}(\text{exp}) \tag{5.56}$$

$$3)\ \frac{|m(K_L) - m(K_S)|}{m(K)} < 7\cdot 10^{-15} m_{K^0}(\text{exp}) \tag{5.57}$$

1) The observed rate for the decay $K_L \to \mu^+\mu^-$ can be understood in terms of QED and the known $K_L \to \gamma\gamma$ transition rate and leaves little room for an elementary $\bar{s}d \to \mu^+\mu^-$ transition.
2) The decay $K^+ \to \pi\nu\bar{\nu}$ can be understood in terms of the elementary reaction $\bar{s} \to \bar{d}\nu\bar{\nu}$.
3) Similarly the smallness of observables linked to $|\Delta S| = 2$ transition amplitudes, such as the $K_L - K_S$ mass difference leaves little room for strangeness changing neutral currents.

Thus, in the Weinberg-Salam model, or more generally in models that allow for neutral current relations that are proportional to the third component of the weak isospin, it is important to prevent the appearance of strangeness changing neutral currents. An elegant solution to the problem of flavour-changing neutral currents was proposed by Glashow, Iliopoulos and Maiani.

We need a "natural mechanism", *i.e.* originating from a symmetry, stable against perturbations, that suppresses 8 orders of magnitude. This can be achieved through the introduction of a fourth quark, the charm quark c. [Glashow, Iliopoulos, Maiani, PRD2(70)1985]

The new multiplet structure is then given by

$$\boxed{\begin{pmatrix} u \\ d_c \end{pmatrix}_L \quad \begin{pmatrix} c \\ s_c \end{pmatrix}_L \quad \begin{matrix} u_R \\ d_{cR} \end{matrix} \quad \begin{matrix} c_R \\ s_{cR} \end{matrix} \qquad (5.58)}$$

(a) The isovector current now reads:

$$j^3_\mu = \sum_{doublets} \bar{Q}_L \gamma_\mu \frac{1}{2} \tau^3 Q_L = \frac{1}{2}[\bar{u}_L \gamma_\mu u_L - \bar{d}_L \gamma_\mu d_L + \bar{c}_L \gamma_\mu c_L - \bar{s}_L \gamma_\mu s_L] \tag{5.59}$$

The addition of the charm quark c diagonalizes the neutral current (*GIM mechanism*) and eliminates $\Delta S \neq 0$, NC reactions.

(b) The electromagnetic current is given by:

$$j^{em}_\mu = \frac{2}{3}[\bar{u}\gamma_\mu u + \bar{c}\gamma_\mu c] - \frac{1}{3}[\bar{d}\gamma_\mu d + \bar{s}\gamma_\mu s] \tag{5.60}$$

(c) The charged current reads:

$$j^-_\mu = \bar{u}\gamma_\mu \frac{1}{2}(1-\gamma_5)[\cos\theta_c d + \sin\theta_c s] + \bar{c}\gamma_\mu \frac{1}{2}(1-\gamma_5)[-\sin\theta_c d + \cos\theta_c s] \tag{5.61}$$

The first term is the Cabibbo current, the second the charm current with strong (c, s) coupling.

In 1973 (1 year before the discovery of the charm quark!) Kobayashi and Maskawa extended Cabibbo's idea to six quarks. We thereby obtain a 3×3 matrix that mixes the weak quarks and the strong quarks. Only in this way the CP violation can be explained. (We come back to this point later.) We also need the 3rd quark family to obtain an anomaly-free theory. We call anomalies terms that violate the classical conservation laws. Thus it can happen that a (classical) local conservation law derived from gauge invariance with the help of Noether's theorem holds at tree level but is not respected by loop diagrams. The simplest example of a Feynman diagram leading to an anomaly is a fermion loop coupled to two vector currents and one axial current. Because the weak interaction contains both vector and axial vector currents there is a danger that such diagrams may arise in the Weinberg-Salam theory and destroy the renormalizability of the theory. The anomaly is canceled if for each lepton doublet we introduce three quark doublets corresponding to the three quark coulours. Since we have three lepton doublets we need to introduce a third quark doublet (with three colours). This was also supported by the observation of a fifth quark (the b quark) in the Υ family.

5.7 The CKM Matrix

5.7.1 Die Fermion Yang-Mills Lagrangian

If we take the down-type quarks in the current basis, then the matrix for the weak interaction of the fermions is diagonal (see also Eqs. (5.57) and (5.61)). With the definitions

$$
\begin{aligned}
U &= \begin{pmatrix} u \\ c \\ t \end{pmatrix} \qquad D' = \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} \\
E &= \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} \qquad N_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} ,
\end{aligned} \tag{5.61}
$$

where $'$ denotes the fields in the current basis, we obtain for the Yang-Mills Lagrangian

$$
\begin{aligned}
\mathcal{L}_{YM-F} &= (\bar{U}_L, \bar{D'}_L) i\gamma^\mu (\partial_\mu + igW_\mu^a \frac{\tau^a}{2} + ig' Y_L B_\mu) \begin{pmatrix} U_L \\ D'_L \end{pmatrix} \\
&+ (\bar{N}_L, \bar{E}_L) i\gamma^\mu (\partial_\mu + igW_\mu^a \frac{\tau^a}{2} + ig' Y_L B_\mu) \begin{pmatrix} N_L \\ E_L \end{pmatrix} \\
&+ \sum_{\Psi_R = U_R, D'_R, E_R} \bar{\Psi}_R i\gamma^\mu (\partial_\mu + ig' Y_R B_\mu) \Psi_R \\
&= \bar{U} i\partial\!\!\!/ U + \bar{D}' i\partial\!\!\!/ D' + \bar{E} i\partial\!\!\!/ E + \bar{N}_L i\partial\!\!\!/ N_L + \mathcal{L}_{int} .
\end{aligned} \tag{5.59}
$$

The interaction Lagrangian reads

$$
\mathcal{L}_{int} = -eJ^\mu_{em} A_\mu - \frac{e}{\sin\theta_W \cos\theta_W} J^\mu_{NC} Z_\mu - \frac{e}{\sqrt{2}\sin\theta_W} (J^{-\mu} W^+_\mu + h.c.) . \tag{5.60}
$$

The electromagnetic current is given by

$$
J^\mu_{em} = Q_u \bar{U} \gamma^\mu U + Q_d \bar{D}' \gamma^\mu D' + Q_e \bar{E} \gamma^\mu E , \tag{5.61}
$$

the neutral weak current by

$$
\begin{aligned}
J^\mu_{NC} &= (\bar{U}_L, \bar{D'}_L) \gamma^\mu \frac{\tau_3}{2} \begin{pmatrix} U_L \\ D'_L \end{pmatrix} + (\bar{N}_L, \bar{E}_L) \gamma^\mu \frac{\tau_3}{2} \begin{pmatrix} N_L \\ E_L \end{pmatrix} - \sin^2\theta_W J^\mu_{em} \\
&= \frac{1}{2} \bar{U}_L \gamma^\mu U_L - \frac{1}{2} \bar{D'}_L \gamma^\mu D'_L + \frac{1}{2} \bar{N}_L \gamma^\mu N_L - \frac{1}{2} \bar{E}_L \gamma^\mu E_L - \sin^2\theta_W J^\mu_{em}
\end{aligned} \tag{5.61}
$$

and the charged weak current by

$$
\begin{aligned}
J^{-\mu} &= (\bar{U}_L, \bar{D'}_L) \gamma^\mu \frac{\tau_1 + i\tau_2}{2} \begin{pmatrix} U_L \\ D'_L \end{pmatrix} + (\bar{N}_L, \bar{E}_L) \gamma^\mu \frac{\tau_1 + i\tau_2}{2} \begin{pmatrix} N_L \\ E_L \end{pmatrix} \\
&= \bar{U}_L \gamma^\mu D'_L + \bar{N}_L \gamma^\mu E_L .
\end{aligned} \tag{5.61}
$$

(The latter is purely left-handed and diagonal in generation space.)

5.7.2 Mass Matrix and CKM Matrix

Remark: Be χ_1, χ_2 $SU(2)$ doublets. Then there are two possibilities to form an $SU(2)$ singlet:
1) $\chi_1^\dagger \chi_2$ and $\chi_2^\dagger \chi_1$
2) $\chi_1^T \epsilon \chi_2$ and $\chi_2^T \epsilon \chi_1$, where

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Proof: Perform an $SU(2)$ transformation

$$\begin{array}{lll} \chi_1(x) & \rightarrow & U(x)\chi_1(x) \qquad \chi_1^\dagger \rightarrow \chi_1^\dagger U^{-1} \\ \chi_2(x) & \rightarrow & U(x)\chi_2(x) \qquad \chi_2^\dagger \rightarrow \chi_2^\dagger U^{-1} , \end{array} \tag{5.60}$$

where

$$U(x) = e^{i\omega_a(x)\tau^a/2} . \tag{5.61}$$

1) is invariant under this transformation.
2) Here we have

$$(U\chi_1)^T \epsilon U \chi_2 = \chi_1^T U^T \epsilon U \chi_2 = \chi_1^T \epsilon \chi_2 \tag{5.62}$$

because with

$$U = e^{iA} = \sum_0^\infty \frac{(iA)^n}{n!} \Rightarrow U^T = \sum_n \frac{(iA^T)^n}{n!} \ , \quad A = \omega_a(x)\frac{\tau^a}{2} . \tag{5.63}$$

And since $(\tau^a)^T \epsilon = -\epsilon \tau^a$, we obtain

$$U^T \epsilon U = \epsilon U^{-1} U = \epsilon , \tag{5.64}$$

so that also 2) is invariant.

The Yukawa Lagrangian: We write up the most general, renormalizable, $SU(2)_L \times U(1)_Y$ invariant Hermitean fermion-fermion-boson Lagrangian. With the $SU(2)$ doublets

$$\begin{pmatrix} U_L \\ D'_L \end{pmatrix} , \begin{pmatrix} N_L \\ E_L \end{pmatrix} , \Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{5.65}$$

and the $SU(2)$ singlets

$$U_R \, , D'_R \, , E_R \tag{5.66}$$

we can construct 2 $SU(2)$ invariant interactions,

$$\Phi^\dagger \begin{pmatrix} \psi_{1L} \\ \psi_{2L} \end{pmatrix} = (\phi^+)^* \psi_{1L} + (\phi^0)^* \psi_{2L} \tag{5.67}$$

and

$$\Phi^T \epsilon \begin{pmatrix} \psi_{1L} \\ \psi_{2L} \end{pmatrix} = \phi^+ \psi_{2L} - \phi^0 \psi_{1L} , \tag{5.68}$$

so that for the Yukawa Lagrangian that conserves also the hypercharge we obtain:

$$\mathcal{L}_{Yuk} = -(\bar{e}_R, \bar{\mu}_R, \bar{\tau}_R) C_E \begin{pmatrix} \Phi^\dagger \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \\ \Phi^\dagger \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix} \\ \Phi^\dagger \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix} \end{pmatrix} + (\bar{u}_R, \bar{c}_R, \bar{t}_R) C_U \begin{pmatrix} \Phi^T \epsilon \begin{pmatrix} u_L \\ d'_L \end{pmatrix} \\ \Phi^T \epsilon \begin{pmatrix} c_L \\ s'_L \end{pmatrix} \\ \Phi^T \epsilon \begin{pmatrix} t_L \\ b'_L \end{pmatrix} \end{pmatrix}$$

$$-(\bar{d}'_R, \bar{s}'_R, \bar{b}'_R) C_D \begin{pmatrix} \Phi^\dagger \begin{pmatrix} u_L \\ d'_L \end{pmatrix} \\ \Phi^\dagger \begin{pmatrix} c_L \\ s'_L \end{pmatrix} \\ \Phi^\dagger \begin{pmatrix} t_L \\ b'_L \end{pmatrix} \end{pmatrix} + h.c. \; . \tag{5.68}$$

The C_E, C_U, C_D are arbitrary complex matrices. We perform through the following unitary transformations a transition into an equivalent field basis (Fields are no observables!)

$$\begin{array}{lll} N_L(x) \rightarrow V_1 N_L(x) & & U_L(x) \rightarrow V_2 U_L(x) \\ E_L(x) \rightarrow V_1 E_L(x) & & D'_L(x) \rightarrow V_2 D'_L(x) \\ E_R(x) \rightarrow U_1 E_R(x) & & U_R(x) \rightarrow U_2 U_R(x) \\ & & D'_R(x) \rightarrow U_3 D'_R(x) \; , \end{array} \tag{5.66}$$

where U_1, U_2, U_3, V_1, V_2 are unitary 3×3 matrices. Since the lepton and quark doublets transform in the same way this does not change the Yang-Mills-, the Higgs- and the Yang-Mills fermion Lagrangian. Only the C matrices are changed:

$$C_E \rightarrow U_1^\dagger C_E V_1 \qquad C_U \rightarrow U_2^\dagger C_U V_2 \qquad C_D \rightarrow U_3^\dagger C_D V_2 \; . \tag{5.67}$$

By choosing the $U_1^\dagger$ and V_1 matrices appropriately we can diagonalize C_E,

$$U_1^\dagger C_E V_1 = \begin{pmatrix} h_e & & \\ & h_\mu & \\ & & h_\tau \end{pmatrix} \qquad \text{with} \quad h_e, h_\mu, h_\tau \geq 0 \; . \tag{5.68}$$

Similarly,

$$U_2^\dagger C_U V_2 = \begin{pmatrix} h_u & & \\ & h_c & \\ & & h_t \end{pmatrix} \qquad \text{with} \quad h_u, h_c, h_t \geq 0 \; . \tag{5.69}$$

Eq. (5.69) fixes the matrix V_2. By choosing U_3 appropriately we obtain

$$U_3^\dagger C_D V_2 = \begin{pmatrix} h_d & & \\ & h_s & \\ & & h_b \end{pmatrix} V^\dagger \qquad \text{with} \quad h_u, h_c, h_t \geq 0 \; . \tag{5.70}$$

where $V^\dagger$ is a unitary matrix. We transform D'_R by $D'_R \rightarrow V^\dagger D'_R$ and obtain

$$C_D \rightarrow V \begin{pmatrix} h_d & & \\ & h_s & \\ & & h_b \end{pmatrix} V^\dagger \; . \tag{5.71}$$

We expand Φ around the vacuum expectation value

$$\Phi = \begin{pmatrix} 0 \\ \frac{v+H(x)}{\sqrt{2}} \end{pmatrix} \tag{5.72}$$

where $H(x)$ is a real field, and obtain

$$\begin{aligned} & (\bar{d}'_R, \bar{s}'_R, \bar{b}'_R) V \begin{pmatrix} h_d & & \\ & h_s & \\ & & h_b \end{pmatrix} V^\dagger \begin{pmatrix} \Phi^\dagger \begin{pmatrix} u_L \\ d'_L \end{pmatrix} \\ \Phi^\dagger \begin{pmatrix} c_L \\ s'_L \end{pmatrix} \\ \Phi^\dagger \begin{pmatrix} t_L \\ b'_L \end{pmatrix} \end{pmatrix} \\ = \; & (\bar{d}'_R, \bar{s}'_R, \bar{b}'_R) V \begin{pmatrix} h_d & & \\ & h_s & \\ & & h_b \end{pmatrix} V^\dagger \begin{pmatrix} \frac{1}{\sqrt{2}}(v+H(x))d'_L \\ \frac{1}{\sqrt{2}}(v+H(x))s'_L \\ \frac{1}{\sqrt{2}}(v+H(x))b'_L \end{pmatrix} . \end{aligned} \tag{5.72}$$

After a basis transformation

$$\begin{pmatrix} d \\ s \\ b \end{pmatrix} = V^\dagger \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} \tag{5.73}$$

we finally have

$$(\bar{d}_R, \bar{s}_R, \bar{b}_R) \begin{pmatrix} h_d & & \\ & h_s & \\ & & h_b \end{pmatrix} \frac{1}{\sqrt{2}}(v+H(x)) \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} . \tag{5.74}$$

The Yang-Mills and the Higgs Lagrangian do not change under the transformation (5.73). But the Yang-Mills fermion Lagrangian becomes

$$\begin{aligned} \mathcal{L}_{YM-F} \;=\; & \bar{U} i\partial\!\!\!/ U + \bar{D} i\partial\!\!\!/ D + \bar{E} i\partial\!\!\!/ E + \bar{N}_L i\partial\!\!\!/ N_L - e J^\mu_{em} A_\mu \\ & - \frac{e}{\sin\theta_W \cos\theta_W} J^\mu_{NC} Z_\mu - \frac{e}{\sqrt{2}\sin\theta_W} (J^{-\mu} W^+_\mu + \text{ h.c.}) . \end{aligned} \tag{5.74}$$

with

$$J^{-\mu} = \bar{U}_L \gamma^\mu D'_L + \bar{N}_L \gamma^\mu E_L = \bar{U}_L \gamma^\mu V D_L + \bar{N}_L \gamma^\mu E_L . \tag{5.75}$$

The unitary 3×3 matrix V is called CKM (Cabibbo-Kobayashi-Maskawa) mixing matrix.

The matrix V is unitary, *i.e.* $V^\dagger V = V V^\dagger = 1$. We investigate the number of free parameters. For a complex $n \times n$ matrix we have $2n^2$ free parameters. Since the matrix is unitary, the number of free parameters is reduced by n^2 equations. Futhermore the phases can be absorbed by a redefinition of the fermion fields, so that the number of free parameters is reduced by further$(2n-1)$ conditions:

Parameters:	$n \times n$ complex matrix:	$2n^2$
	unitarity:	n^2
	free phase choice:	$2n-1$
		$(n-1)^2$ free parameters

In the Euler parametrisation we have

rotation angles: $\frac{1}{2}n(n-1)$
phases: $\frac{1}{2}(n-1)(n-2)$

Thus we find for $n = 2, 3$

n	angles	phases
2	1	0
3	3	1

We thereby find that in a

2 − family theory ∼ Cabibbo: no $\mathcal{CP}$ violation with L currents
3 − Familien Theorie ∼ KM: complex matrix → $\mathcal{CP}$ violation
"Prediction of a 3-family structure"

Next we investigate how we can parametrise the matrix:

(i) Esthetic parametrisation:

$$V_{CKM} = R_{sb}(\theta_2)U(\delta)R_{sd}(\theta_1)R_{sb}(\theta_3) \tag{5.74}$$

with

$$\begin{aligned} 0 &\leq \theta_i \leq \pi/2 \\ -\pi &\leq \delta \leq +\pi \end{aligned} \tag{5.74}$$

and

$$R_{sb}(\theta_2) = \begin{pmatrix} 1 & 0 & \\ 0 & \cos\theta_2 & \sin\theta_2 \\ 0 & -\sin\theta_2 & \cos\theta_2 \end{pmatrix} \quad \text{etc.} \quad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \tag{5.75}$$

(ii) Convenient parametrisation (Wolfenstein):

$$V = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} \tag{5.76}$$

The determination of the parameters is done by

(a) Cabibbo theory: $\lambda = 0.221 \pm 0.002$
(b) $b \to c$ decays: $V_{cb} = A\lambda^2$ $\to A = 0.78 \pm 0.06$
(c) $b \to u$ decays: $|V_{ub}/V_{cb}| = 0.08 \pm 0.02$ $\to (\rho^2 + \eta^2)^{1/2} = 0.36 \pm 0.09$
(d) t matrix elements through unitarity

(e) $\mathcal{CP}$ violation:

The unitarity of the CKM matrix leads to the unitarity triangle

$$\begin{aligned} V_{ud}^* V_{td} + V_{us}^* V_{ts} + V_{ub}^* V_{tb} &= 0 \\ A\lambda^3(1-\rho-i\eta) - A\lambda^3 + A\lambda^3(\rho+i\eta) &= 0 \\ \Rightarrow (\rho + i\eta) + (1-\rho-i\eta) &= 1 \end{aligned} \tag{5.75}$$

We hence have the unitarity triangle

with the edges $(0,0)$, $(1,0)$, (ρ,η) (in the complex plane) and the angles α, β, γ. The determination is done through

(*i*) $\rho^2 + \eta^2$, circle around 0, from $b \to u$ and $b \to c$ decays.

(*ii*) $\eta > 0$ from the $\mathcal{CP}$ violation in the K system.

(*iii*) $B_d - \bar{B}_d$ oscillations:

$|1 - \rho - i\eta| = 1.03 \pm 0.22$

(*iv*) β from $\mathcal{CP}$ violation in $B \to J/\Psi K$
α from $\mathcal{CP}$ violation in $B \to \pi\pi$
γ from $\mathcal{CP}$ violation in $B \to \rho K$

Kapitel 6

The Standard Model Higgs Sector

<u>*Literature*</u>:

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6.1 The Introduction of the Higgs Boson

There are two reasons for the introduction of the Higgs boson in the Standard Model (SM) of particle physics:

1. A theory of massive gauge bosons and fermions, which is weakly interacting up to very high energies, requires for unitarity reasons the existence of a Higgs particle. The Higgs particle is a scalar 0^+ particle, *i.e.* a spin 0 particle with positive parity, which couples to the other particles with a coupling strength proportional to the mass (squared) of the particles.
   Look *e.g.* at the amplitude for the scattering of longitudinal gauge bosons W_L into a pair of longitudinal gauge bosons W_L, see Fig. 6.1. Without a Higgs boson the amplitude diverges proportional to the center-of-mass (c.m) energy squared, s, *cf.* Fig. 6.1 (upper), where G_F denotes the Fermi constant. The introduction of a Higgs boson which couples proportional to the mass squared of the gauge boson, regularizes the amplitude, *cf.* Fig. 6.1 (lower), where M_H denotes the Higgs boson mass.

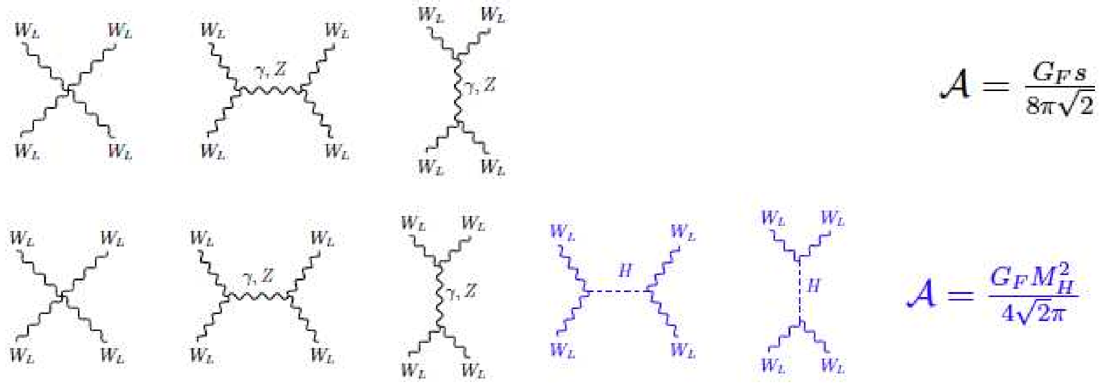


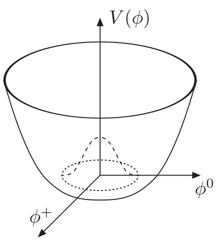
Abbildung 6.1: The scattering of longitudinal gauge bosons in longitudinal gauge bosons. Upper: without a Higgs boson. Lower: with a Higgs boson

2. The introduction of mass terms for the gauge bosons violates the $SU(2)_L \times U(1)$ symmetry of the SM Lagrangian. The same problem arises for the introduction of mass terms for the fermions.

6.2 The Standard Model Higgs sector

The problem of mass generation without violating gauge symmetries can be solved by introducing an $SU(2)_L$ Higgs doublet Φ with weak isospin $I = 1/2$ and hypercharge $Y = 1$ and the SM Higgs potential given by

$$V(\Phi) = \lambda[\Phi^\dagger \Phi - \frac{v^2}{2}]^2 \ . \tag{6.1}$$



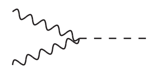
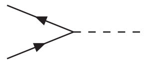
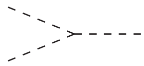
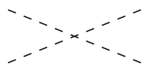
Introducing the Higgs field in a physical gauge,

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H \end{pmatrix} , \tag{6.2}$$

the Higgs potential can be written as

$$V(H) = \frac{1}{2} M_H^2 H^2 + \frac{M_H^2}{2v} H^3 + \frac{M_H^2}{8v^2} H^4 \ . \tag{6.3}$$

Here we can read off directly the mass of the Higgs boson and the Higgs trilinear and quartic self-interactions. Adding the couplings to gauge bosons and fermions we have

$\mathcal{M}$ass of the Higgs boson	$M_H = \sqrt{2\lambda}v$	
$\mathcal{C}$ouplings to gauge bosons	$g_{VVH} = \frac{2M_V^2}{v}$	
$\mathcal{Y}$ukawa couplings	$g_{ffH} = \frac{m_f}{v}$	
$\mathcal{T}$rilinear coupling [units $\lambda_0 = 33.8$ GeV]	$\lambda_{HHH} = 3\frac{M_H^2}{M_Z^2}$	
$\mathcal{Q}$uartic coupling [units λ_0^2]	$\lambda_{HHHH} = 3\frac{M_H^2}{M_Z^4}$	

In the SM the trilinear and quartic Higgs couplings are uniquely determined by the mass of the Higgs boson.

The Higgs potential with its typical form leads to a non-vanishing vacuum expectation value (VEV) v in the ground state

$$v = \frac{1}{\sqrt{\sqrt{2}G_F}} \approx 246 \text{ GeV}. \tag{6.4}$$

Expansion of Φ around the minimum of the Higgs potential leads to one massive scalar particle, the Higgs boson, and three massless Goldstone bosons, that are absorbed to give masses to the charged W bosons and the Z boson. (For a toy example, see Appendix **??**.) The appearance of Goldstone bosons is stated in the Goldstone theorem, which says:

Be

N = dimension of the algebra of the symmetry group of the complete Lagrangian.
M = dimension of the algebra of the group, under which the vacuum is invariant after spontaneous symmetry breaking.

$\Rightarrow$ There are N-M Goldstone bosons without mass in the theory.

The Goldstone theorem states, that for each spontaneously broken degree of freedom of the symmetry there is one massless Goldstone boson.

In gauge theories, however, the conditions for the Goldstone theorem are not fulfilled: Massless scalar degrees of freedom are absorbed by the gauge bosons to give them mass. The Goldstone phenomenon leads to the Higgs phenomenon.

6.3 Verification of the Higgs mechanism

On the 4th July 2012, the LHC experiments ATLAS and CMS announced the disovery of a new scalar particle with mass $M_H \approx 125$ GeV. The discovery triggered immediately the investigation of the properties of this particle in order to test if it is indeed the Higgs particle, that has been disovered. In order to verify experimentally the Higgs mechanism as the mechanism which allows to generate particle masses without violating gauge principles, we have to perform several steps:

1.) First of all the Higgs particle has to be discovered.

2.) In the next step its couplings to gauge bosons and fermions are measured. If the Higgs mechanism acts in nature these couplings are proportional to the masses (squared) of the respective particles.

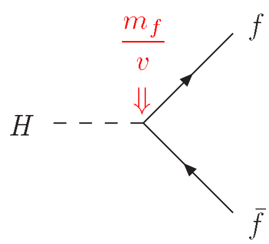
3.) Its spin and parity quantum numbers have to be determined.

4.) And finally, the Higgs trilinear and quartic self-couplings must be measured. This way, the Higgs potential can be reconstructed which, with its typical minimax form, is responsible for the non-vanishing vacuum expectation value, that is essential for the non-zero particle masses.

In the following, we will see how this program can be performed at the hadron collider LHC.

6.4 Higgs boson decays

In order to search for the Higgs boson at existing and future colliders, one has to know what to look for. Hence, one has to study the Higgs decay channels. Since the Higgs boson couples proportional to the mass of the particle its preferred decays will be those into heavy particles, *i.e.* heavy fermions and, when kinematically allowed, into gauge bosons. The branching ratios into fermions are for $M_H = 125.09$ GeV[1]



$$\begin{array}{lcl} BR(H \to b\bar{b}) & = & 0.5797 \\ BR(H \to \tau^+\tau^-) & = & 0.06245 \\ BR(H \to c\bar{c}) & = & 0.02879 \\ BR(H \to t\bar{t}) & = & 0 \end{array} . \tag{6.4}$$

They are obtained from the partial width $\Gamma(H \to f\bar{f})$ into fermions and the total width Γ_{tot}, which is given by the sum of all partial decay widths of the Higgs boson,

$$BR(H \to f\bar{f}) = \frac{\Gamma(H \to f\bar{f})}{\Gamma_{\text{tot}}} . \tag{6.5}$$

The tree-level partical decay width into fermions is given by

$$\Gamma(H \to f\bar{f}) = \frac{N_{cf} G_F M_H}{4\sqrt{2}\pi} m_f^2 \beta^3 , \tag{6.6}$$

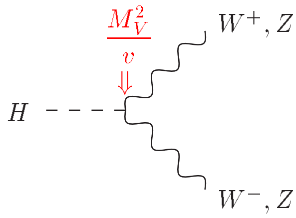
with the velocity

$$\beta = (1 - 4m_f^2/M_H^2)^{1/2} \tag{6.7}$$

of the fermions, their mass m_f, and the colour factor $N_{cf} = 1(3)$ for leptons (quarks). These decays receive large QCD corrections which have been calculated by various groups and can amount up to -50%. The electroweak corrections in total are small of $\mathcal{O}(5\%)$, in the intermediate mass range.

The branching ratios into massive gauge bosons reach for $M_H = 125.09$ GeV

[1] These decays have been calculated with `HDECAY`, a Fortran code by A. Djouadi, J. Kaliniowski, M. Muhlleitner and M. Spira. It can be downloaded at: http://tiger.web.psi.ch/hdecay



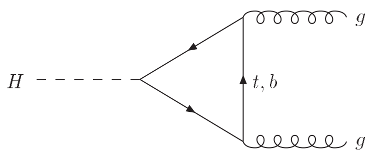
$$\begin{array}{lcl} BR(H \to W^+W^-) & = & 0.2167 \\ BR(H \to ZZ) & = & 0.02656 \end{array} . \tag{6.8}$$

The tree-level decay width into a pair of on-shell massive gauge bosons $V = Z, W$ is given by

$$\Gamma(H \to VV) = \delta_V \frac{G_F M_H^3}{16\sqrt{2}\pi} \beta(1 - 4x + 12x^2) , \tag{6.9}$$

with $x = M_V^2/M_H^2$, $\beta = \sqrt{1-4x}$ and $\delta_V = 2(1)$ for $V = W(Z)$. The electroweak corrections to these decays are of the order 5-20%. For a Higgs boson of mass $M_H = 125$ GeV off-shell decays $H \to V^*V^* \to 4l$ are important. The program `PROPHECY4F` includes the complete QCD and EW next-to-leading order (NLO) corrections to $H \to WW/ZZ \to 4f$.

The decay into gluon pairs proceeds via a loop with the dominant contributions from top and bottom quarks and for $M_H = 125.09$ GeV has a branching ratio of:



$$BR(H \to gg) = 0.08157 . \tag{6.10}$$

At leading order (LO) the decay width can be cast into the form

$$\Gamma_{LO}(H \to gg) = \frac{G_F \alpha_s^2 M_H^3}{36\sqrt{2}\pi^3} \left| \sum_{Q=t,b} A_Q^H(\tau_Q) \right|^2 , \tag{6.11}$$

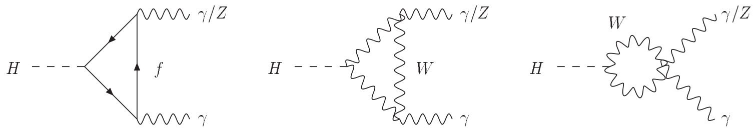
with the form factor

$$A_Q^H = \frac{3}{2}\tau[1 + (1-\tau)f(\tau)] \tag{6.12}$$

$$f(\tau) = \begin{cases} \arcsin^2 \frac{1}{\sqrt{\tau}} & \tau \geq 1 \\ -\frac{1}{4}\left[\log \frac{1+\sqrt{1-\tau}}{1-\sqrt{1-\tau}} - i\pi\right]^2 & \tau < 1 \end{cases} \tag{6.13}$$

The parameter $\tau_Q = 4M_Q^2/M_H^2$ is defined by the pole mass M_Q of the heavy loop quark Q. Note that for large quark masses the form factor approaches unity. The strong coupling constant is denoted by α_s. The QCD corrections have been calculated. They are large and increase the branching ratio by about 70% at next-to-leading order (NLO). They are known up to next-to-next-to-next-to leading order(N^3LO). The electroweak corrections increase the partial Higgs decay width into gluons by about 5%.

Further loop-mediated decays are those into 2 photons or a photon and a Z boson. They are mediated by charged fermion and W boson loops, the latter being dominant.



Although (for $M_H = 125.09$ GeV) they amount only up to

$$BR(H \to \gamma\gamma) = 2.265 \times 10^{-3} \quad (6.14)$$
$$BR(H \to Z\gamma) = 1.537 \times 10^{-3} \quad (6.15)$$

the $\gamma\gamma$ final state is an important search mode for light Higgs bosons at the LHC. The partial decay width into photons reads

$$\Gamma(H \to \gamma\gamma) = \frac{G_F \alpha^2 M_H^3}{128\sqrt{2}\pi^3} \left| \sum_f N_{cf} e_f^2 A_f^H(\tau_f) + A_W^H(\tau_W) \right|^2 , \quad (6.16)$$

with the form factors

$$A_f^H(\tau) = 2\tau[1 + (1-\tau) f(\tau)] \quad (6.17)$$
$$A_W^H(\tau) = -[2 + 3\tau + 3\tau(2-\tau) f(\tau)] , \quad (6.18)$$

with the function $f(\tau)$ defined in Eq. (6.13). The parameters $\tau_i = 4M_i^2/M_H^2$ $(i = f, W)$ are defined by the corresponding masses of the heavy loop particles. N_{cf} denotes again the colour factor of the fermion and e_f the electric charge. For large loop masses the form factors approach constant values,

$$\begin{array}{llll} A_f^H & \to \frac{4}{3} & \text{for } M_H^2 & \ll 4M_Q^2 \\ A_W^H & \to -7 & \text{for } M_H^2 & \ll 4M_W^2 . \end{array} \quad (6.19)$$

The W loop provides the dominant contribution in the intermediate Higgs mass regime, and the fermion loops interfere destructively. The QCD corrections have been calculated and are small in the intermediate Higgs boson mass region. The tree-level decay width into $Z\gamma$ is given

$$\Gamma(H \to Z\gamma) = \frac{G_F^2 M_W^2 \alpha M_H^3}{64\pi^4} \left(1 - \frac{M_Z^2}{M_H^2}\right)^3 \left| \sum_f A_f^H(\tau_f, \lambda_f) + A_W^H(\tau_W, \lambda_W) \right|^2 , \quad (6.20)$$

with the form factors

$$\begin{aligned} A_f^H(\tau,\lambda) &= 2N_{cf} \frac{e_f(I_{3f} - 2e_f \sin^2\theta_W)}{\cos\theta_W} [I_1(\tau,\lambda) - I_2(\tau,\lambda)] \\ A_W^H(\tau,\lambda) &= \cos\theta_W \Big\{ 4(3 - \tan^2\theta_W) I_2(\tau,\lambda) \\ &\quad + \left[\left(1 + \frac{2}{\tau}\right) \tan^2\theta_W - \left(5 + \frac{2}{\tau}\right) \right] I_1(\tau,\lambda) \Big\} . \end{aligned} \quad (6.19)$$

The functions I_1 and I_2 read

$$I_1(\tau,\lambda) = \frac{\tau\lambda}{2(\tau-\lambda)} + \frac{\tau^2\lambda^2}{2(\tau-\lambda)^2}[f(\tau) - f(\lambda)] + \frac{\tau^2\lambda}{(\tau-\lambda)^2}[g(\tau) - g(\lambda)] \quad (6.20)$$
$$I_2(\tau,\lambda) = -\frac{\tau\lambda}{2(\tau-\lambda)}[f(\tau) - f(\lambda)] . \quad (6.21)$$

The function $g(\tau)$ can be cast into the form

$$g(\tau) = \begin{cases} \sqrt{\tau - 1}\arcsin\frac{1}{\sqrt{\tau}} & \tau \geq 1 \\ \frac{\sqrt{1-\tau}}{2}\left[\log\frac{1+\sqrt{1-\tau}}{1-\sqrt{1-\tau}} - i\pi\right] & \tau < 1 \end{cases} \tag{6.22}$$

The parameters $\tau_i = 4M_i^2/M_H^2$ and $\lambda_i = 4M_i^2/M_Z^2$ $(i = f, W)$ are defined in terms of the corresponding masses of the heavy loop particles. The W loop dominates in the intermediate Higgs mass range, and the heavy fermion loops interfere destructively.

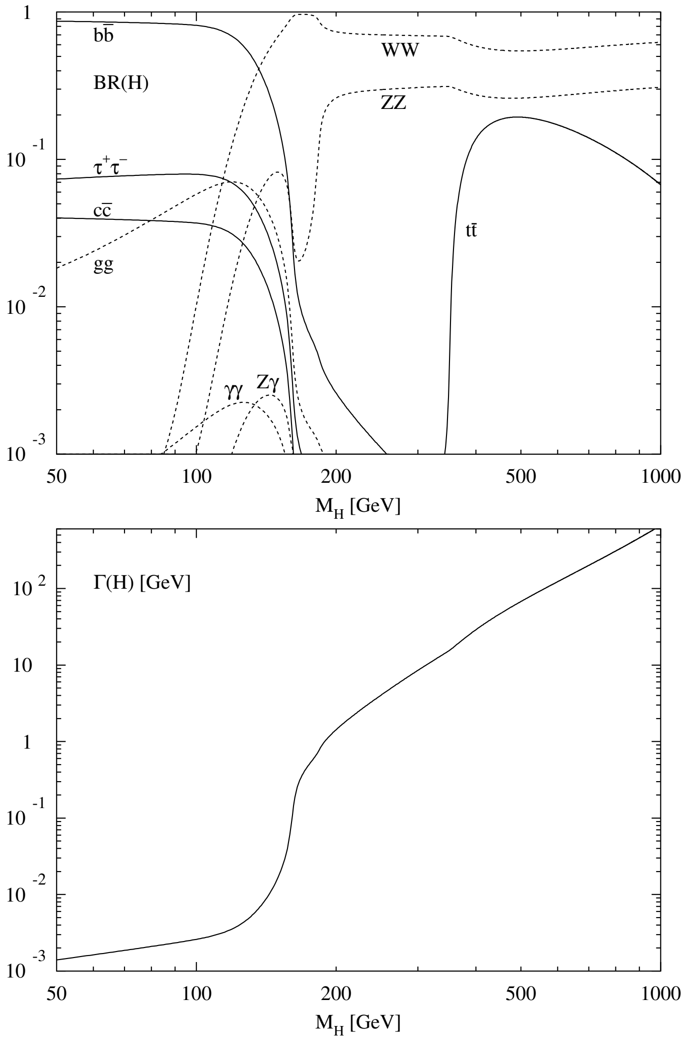


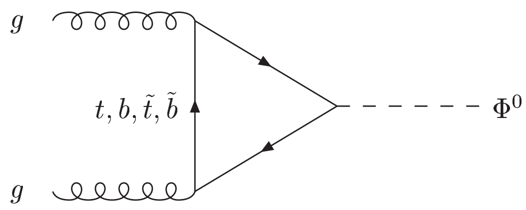
Abbildung 6.2: The Higgs boson branching ratios (upper) and the total width (lower) as a function of the Higgs boson mass. Plote made with `HDECAY`. Now we know that the Higgs boson mass is $M_H = 125.09$ GeV.

Figs.6.2 show the Higgs boson branching ratios and total width as a function of the Higgs boson mass. One can infer from the figures that the total Higgs boson width is rather small, it is $4.108 \cdot 10^{-3}$ GeV, for $M_H = 125.09$ GeV.

6.5 Higgs boson production at the LHC

There are several Higgs boson production mechanisms at the LHC.

- Gluon fusion: The dominant production mechanism for Standard Model Higgs bosons at the LHC is gluon fusion



$$pp \to gg \to H \ . \tag{6.23}$$

In the Standard Model it is mediated by top and bottom quark loops. The QCD corrections (the next-to leading order calculation involves 2-loop diagrams!) have been calculated and turn out to be large. They are of the order 10-100%; see Fig. 6.3, which shows the NLO K-factor, *i.e.* the ratio of the NLO cross section to the leading order (LO) cross section as a function of the Higgs boson mass for the virtual and real corrections.

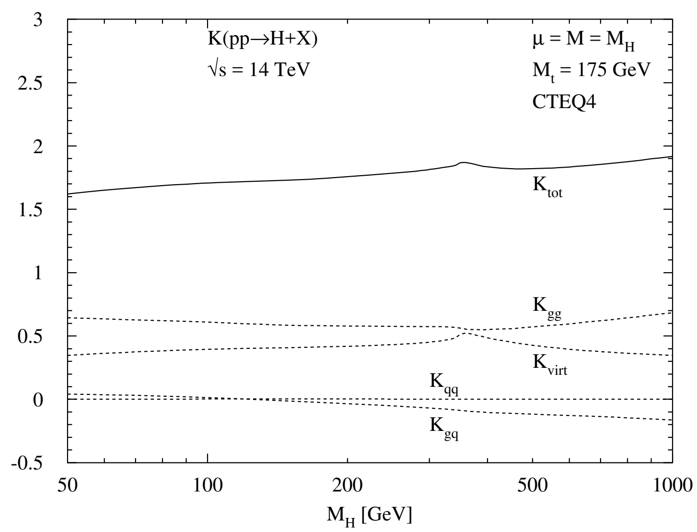


Abbildung 6.3: The K factor for the gluon fusion process as a function of the Higgs boson mass.

Due to the inclusion of the NLO QCD corrections the scale dependence of the gluon fusion cross section is decreased, *cf.* Fig. 6.4.

The next-to-next-to leading order (NNLO) corrections have been calculated in the limit of heavy top quark masses ($M_H \ll m_t$). They lead to a further increase of the cross section by 20-30%. The scale dependence is reduced to $\Delta \lesssim 10 - 15\%$. Resummation of the soft gluons adds another 10%. More recently, also the N^3LO corrections have been calculated in the heavy top quark mass limit. These corrections range at the level of a few percent.

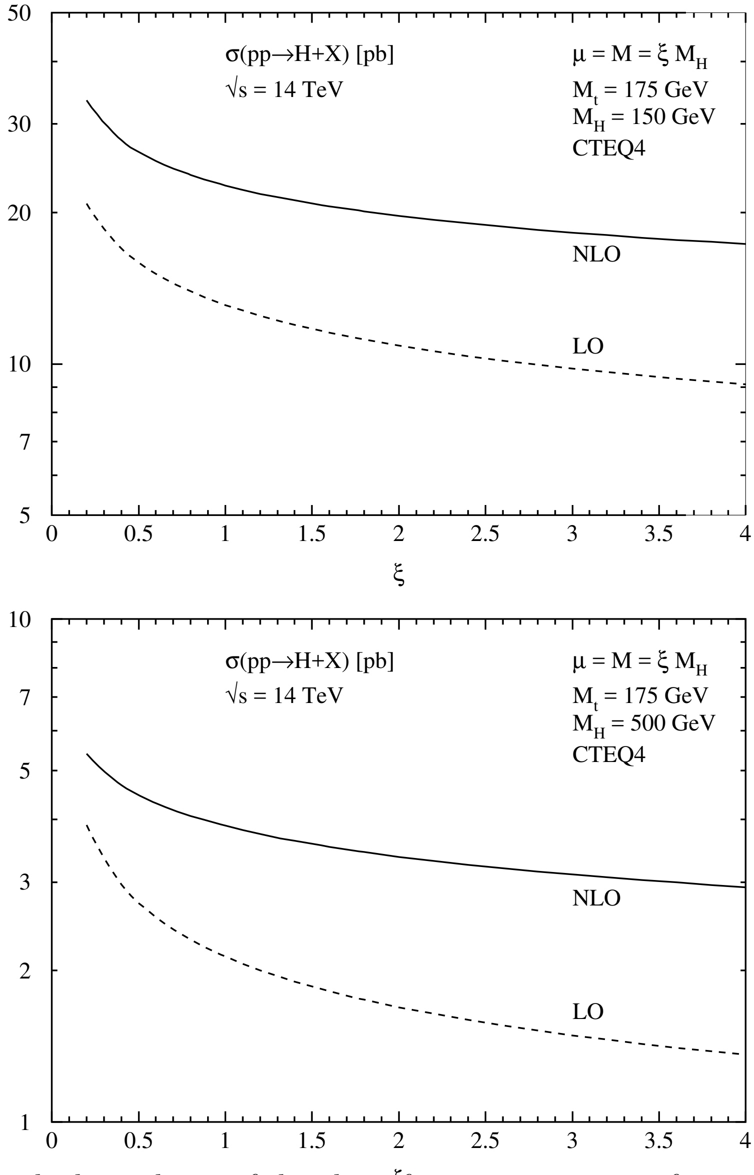
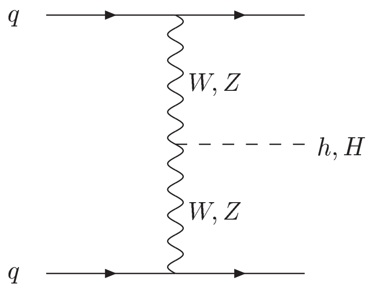


Abbildung 6.4: The scale dependence of the gluon fusion cross section for two different Higgs masses.

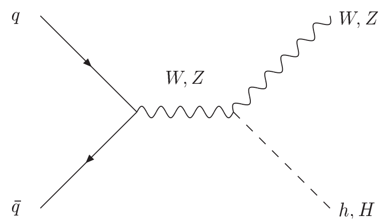
- WW/ZZ fusion: Higgs bosons can be produced in the WW/ZZ fusion processes



$$pp \to W^* W^* / Z^* Z^* \to H \ . \tag{6.24}$$

The QCD corrections have been calculated and amount up to $\sim 10\%$. In the meantime more higher order QCD and EW corrections have been calculated. (Not treated here.)

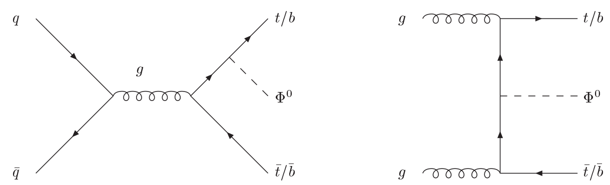
- Higgs-strahlung: Higgs bosons production in Higgs-strahlung [Glashow et al.; Kunszt et al.] proceeds via



$$pp \to W^*/Z^* \to W/Z + H \; . \tag{6.25}$$

The QCD corrections are $\sim$ 30%. The NNLO QCD corrections add another 5-10%. The theoretical error is reduced to about 5%. However, in the ZH final state there is also a sizeable contribution from the process $gg \to ZH$ with about $\sim$ 20% to the total cross section. Recently the NLO QCD corrections to $gg \to ZH$ have been calculated in the heavy-top-quark limit [185]. They increase this contribution significantly.

- Associated Production: Higgs bosons can also be produced in association with top and bottom quarks



$$pp \to t\bar{t}/b\bar{b} + H \; . \tag{6.26}$$

The NLO QCD corrections to associated top production increase the cross section at the LHC by 20%.

For all the production and background processes a lot of progress has been made in the last years on the calculation of the higher order (HO) QCD and EW corrections. They are not subject of this lecture, though. For details, see the corresponding literature.

Fig. 6.5 shows the production cross section in pb as a function of the Higgs boson mass.

6.6 Higgs Boson Discovery

The main Higgs discovery channels are the $\gamma\gamma$ and ZZ^* final states. The decay into $\gamma\gamma$ final states has a very small branching ratio, but is very clean. (CMS and ATLAS have an excellent photon-energy resolution. Look for narrow $\gamma\gamma$ invariant mass peak, extrapolate background into the signal region from thresholds.). The ZZ^* final state is the other important search

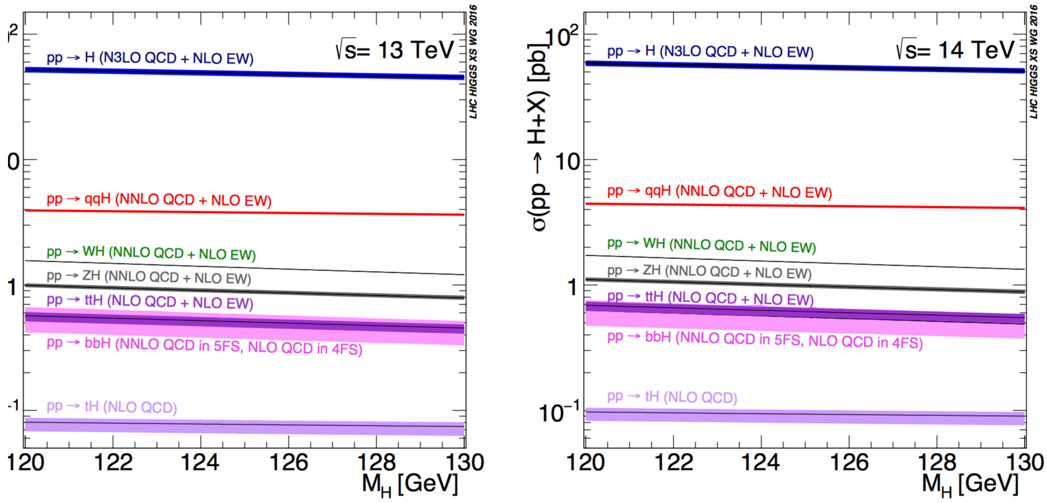


Abbildung 6.5: Higgs boson production cross sections as a function of the Higgs mass for 13 and 14 TeV c.m. energy at the LHC including the most up-to-date higher-order corrections as indicated at the shown cross section bands. The size of the bands reflects the total estimated theoretical uncertainties. From LHC Higgs XS WG report 2016, arXiv:1610.07922.

channel. For $M_H = 125.09$ GeV it is an off-shell decay. It leads to a clean 4 lepton (4l) final state from the decay of the Z bosons. Also the WW final state is off-shell. The final state signature includes missing energy from the neutrinos of the W boson decays. The $b\bar{b}$ final state is exploited as well. It has the largest branching ratio, but suffers from a large QCD background. Finally, the $\tau\tau$ channel is also used.

The experiments give the best fit values to the reduced μ values in the final state X. These are the production rate times branching ratio into the final state $X = \gamma, Z, W, b, \tau$ normalized to the corresponding value for a SM Higgs boson,

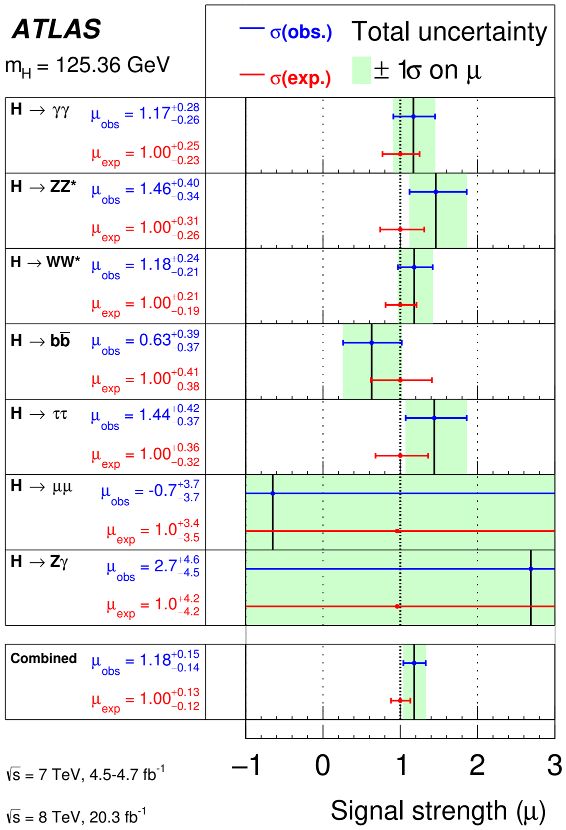
$$\mu = \frac{\sigma_{\text{prod}} \times BR(H \to XX)}{(\sigma_{\text{prod}} \times BR(H \to XX))_{\text{SM}}} \,. \tag{6.27}$$

In case the discovered Higgs boson is a SM Higgs boson they are all equal to 1. Figure 6.6 shows the μ values reported by the LHC experiments. At present the the various final states suffer from uncertainties that leave room for beyond the SM (BSM) physics.

The main discovery channels for the 125 GeV Higgs boson at ATLAS and CMS, *i.e.* the photon and the Z boson final states, are shown in Fig. 6.7.

6.7 Higgs boson couplings at the LHC

In principle the strategy to measure the Higgs boson couplings is to combine various Higgs production and decay channels, from which the couplings can then be extracted. For example, the production of the Higgs boson in W boson fusion with subsequent decay into τ leptons,



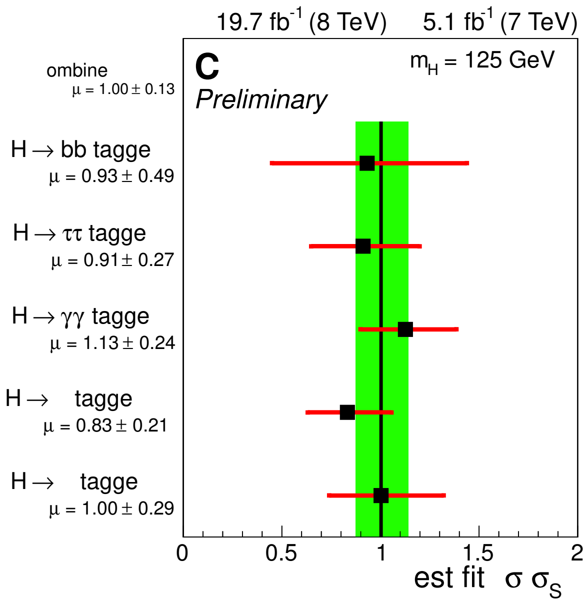


Abbildung 6.6: Best fits for the μ values reported by ATLAS and CMS.

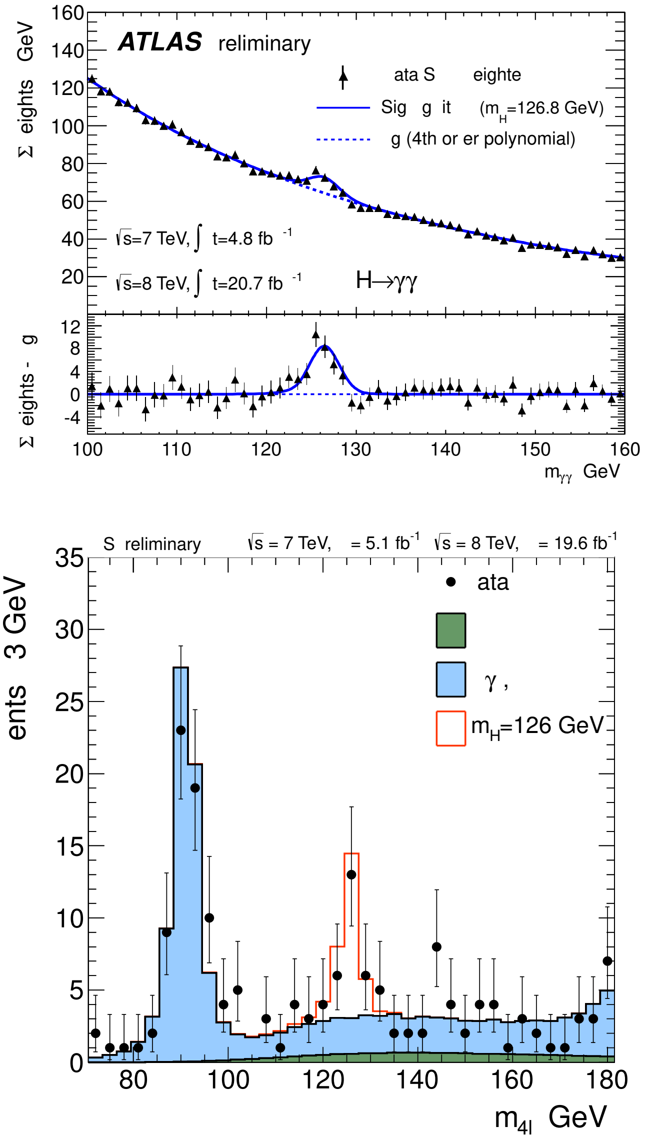


Abbildung 6.7: The main Higgs discovery channels: Upper: The photon final state, here shown for the ATLAS experiment [ATLAS-CONF-2013-12]. Lower: The ZZ^* final state, here shown for the CMS experiment [CMS-PAS-HIG-13-002].

Fig. 6.8, is proportional to the partial width into WW and the branching ratio into $\tau\tau$. Combination with other production/decay channels and the knowledge of the total width allow then to extract the Higgs couplings. The problem at the LHC, however, is that the total width, which is small for a SM 125 GeV Higgs boson, cannot be measured without model-assumptions, and also not all final states are experimentally accessible. Therefore without applying model-assumptions only ratios of couplings are measureable.

The theoretical approach is to define an effective Lagrangian with modified Higgs couplings. In a first approach the couplings are modified by overall scale factors κ_i and the tensor structure is not changed. With this Lagrangian the signal rates, respectively μ values, are calculated as function of the scaling factors, $\mu(\kappa_i)$. These are then fitted to the experimen-

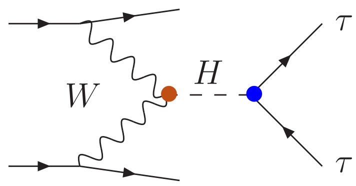


Abbildung 6.8: Feynman diagram for the production of a Higgs boson in W boson fusion with subsequent decay into $\tau\tau$. It is proportional to the partial width Γ_{WW} and the branching ratio into $\tau\tau$, BR$(H \to \tau\tau)$.

tally measured μ values. The fits provide then the κ_i values. Such a theoretical Lagrangian for the SM field content with a scalar particle h looks like

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_h - (M_W^2 W_\mu^+ W^{\mu-} + \frac{1}{2} M_Z^2 Z_\mu Z^\mu)[1 + 2\,\kappa_V\,\frac{h}{v} + \mathcal{O}(h^2)] \\ &\quad - m_{\psi_i} \bar{\psi}_i \psi_i [1 + \kappa_F\,\frac{h}{v} + \mathcal{O}(h^2)] + ... \end{aligned} \quad (6.27)$$

It is valid below the scale Λ where new physics (NP) becomes important. It implements the electroweak symmetry breaking (EWSB) via $\mathcal{L}_h$ and the custodial symmetry through $\kappa_W = \kappa_Z = \kappa_V$. Furthermore, there are no tree-level flavour changing neutral current (FCNC) couplings as κ_F is chosen to be the same for all fermion generations and does not allow for transitions between fermion generations. The best fit values for κ_f and κ_V are shown in Fig. 6.9.

If the discovered particle is the Higgs boson the coupling strengths are proportional to the masses (squared) of the particles to which the Higgs boson couples. This trend can be seen in the plot published by CMS, see Fig. 6.10.

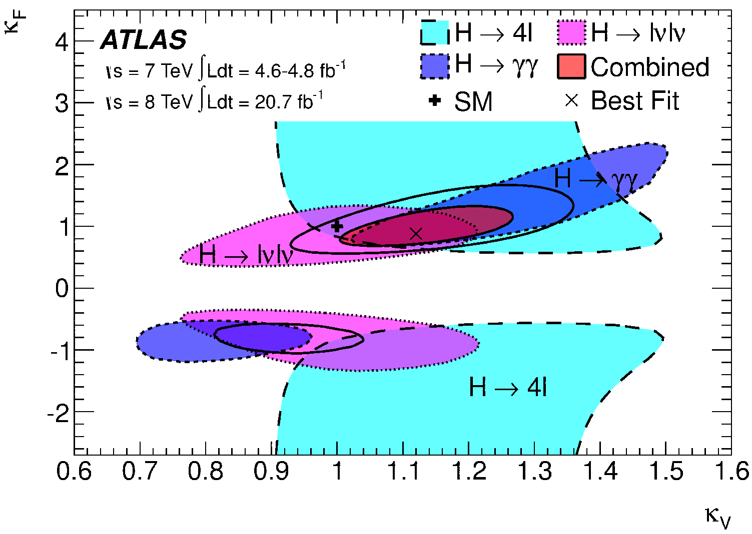
6.8 Higgs Boson Quantum Numbers

The Higgs boson quantum numbers can be extracted by looking at the threshold distributions and the angular distributions of various production and decay processes. The SM Higgs boson has spin 0, positive parity P and is even under charge conjugation C. From the observation of the Higgs boson in the $\gamma\gamma$ final state one can already conclude that it does not have spin 1, due to the Landau-Yang theorem, and that it has $C = +1$, assuming charge invariance. However, these are theoretical considerations and have to be proven also experimentally.

The theoretical tools to provide angular distributions for a particle with arbitrary spin and parity are helicity analyses and operator expansion. Let us look as an example at the Higgs decay into ZZ^*, and the Z bosons subsequently decay into 4 leptons,

$$H \to ZZ^{(*)} \to (f_1\bar{f}_1)(f_2\bar{f}_2)\ . \quad (6.28)$$

The decay is illustrated in Fig. 6.11. The angle φ is the azimuthal angle between the decay planes of the Z bosons in the H rest frame. The θ_1 and θ_2 are the polar angles, respectively, of the fermion pairs in, respectively, the rest frame of the decaying Z boson.



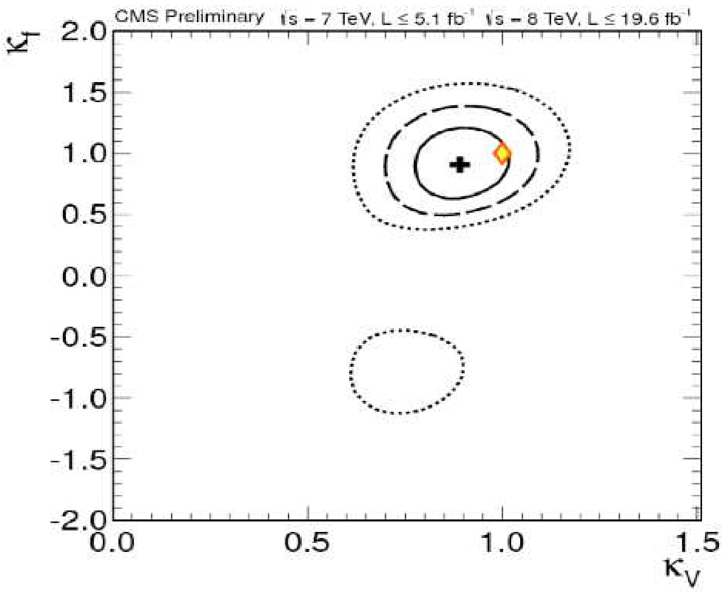


Abbildung 6.9: The best fit values for κ_f and κ_V by ATLAS [Phys. Lett. B726 (2013) 88] (upper) and CMS [CMS-PAS-HIG-13-005] (lower).

For the SM the double polar angular distribution reads

$$\frac{1}{\Gamma'}\frac{d\Gamma'}{d\cos\theta_1 d\cos\theta_2} = \frac{9}{16}\frac{1}{\gamma^4+2}\Big[\gamma^4\sin^2\theta_1\sin^2\theta_2 + \frac{1}{2}(1+\cos^2\theta_1)(1+\cos^2\theta_2)\Big] \tag{6.28}$$

and the azimuthal angular distribution is given by

$$\frac{1}{\Gamma'}\frac{d\Gamma'}{d\phi} = \frac{1}{2\pi}\left[1+\frac{1}{2}\frac{1}{\gamma^4+2}\cos 2\phi\right] \tag{6.29}$$

The verification of these distributions is a necessary step for the proof of the 0^+ nature of the Higgs boson.

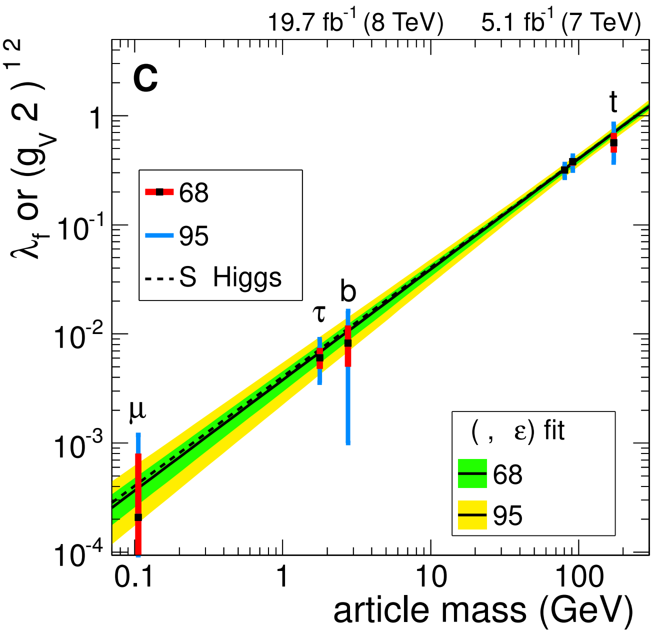


Abbildung 6.10: Coupling strengths as function of the mass of the particles coupled to the Higgs boson, CMS [CMS-PAS-HIG-14-009]

The calculation of the azimuthal angular distribution delivers a different behaviour for a scalar and a pseudoscalar boson:

$$\begin{array}{lll} 0^+ & : & d\Gamma/d\phi \sim 1 + 1/(2\gamma^4 + 4)\cos 2\phi \\ 0^- & : & d\Gamma/d\phi \sim 1 - 1/4\cos 2\phi \end{array} \quad (6.30)$$

Here $\gamma^2 = (M_H^2 - M_*^2 - M_Z^2)/(2M_* M_Z)$ and M_* is the mass of the virtual Z boson. Figure 6.12 shows how the azimuthal angular distribution can be exploited to test the parity of the particle. A pseudoscalar with spin-parity 0^- shows the opposite behaviour in this distribution than the scalar, which is due to the minus sign in front of $\cos 2\phi$ in Eq. (6.30). The threshold behaviour on the other hand can be used to determine the spin of the particle. We have for spin 0 a linear rise with the velocity β,

$$\frac{d\Gamma[H \to Z^* Z]}{dM_*^2} \sim \beta = \sqrt{(M_H - M_Z)^2 - M_*^2}/M_H \; . \quad (6.31)$$

A spin 2 particle, *e.g.* shows a flatter rise, $\sim \beta^3$, *cf.* Fig. 6.13.

The experiments cannot perform an independent spin-parity measurement. Instead they test various spin-parity hypotheses. Various non-SM spin-parity hypotheses have been ruled out at more than 95% confidence level (C.L.), see *e.g.* Figs. 6.14 and 6.15.

6.9 Determination of the Higgs self-interactions

In order to fully establish the Higgs mechanism as the one responsible for the generation of particle masses without violating gauge principles, the Higgs potential has to be reconstructed. This can be done once the Higgs trilinear and quartic self-interactions have been

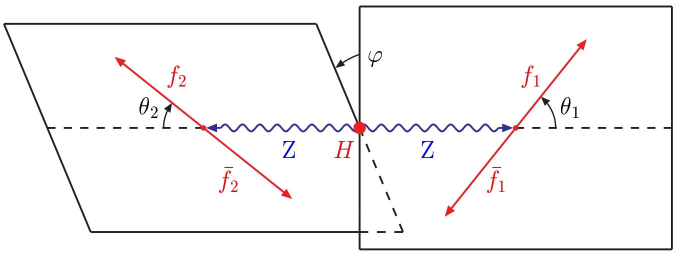


Abbildung 6.11: The decay $H \to ZZ^{(*)} \to (f_1\bar{f}_1)(f_2\bar{f}_2)$.

measured. The trilinear coupling λ_{HHH} is accessible in double Higgs production. The quartic coupling λ_{HHHH} is to be obtained from triple Higgs production.

6.9.1 Determination of the Higgs self-couplings at the LHC

The processes for the extraction of λ_{HHH} at the LHC are gluon fusion into a Higgs pair, double Higgs strahlung, double WW/ZZ fusion and radiation of a Higgs pair off top quarks.

$$\begin{array}{llcllcl} \text{gluon fusion:} & gg & \to & HH & & & \\ \text{double Higgs-strahlung:} & q\bar{q} & \to & W^*/Z^* & \to & W/Z + HH & \\ WW/ZZ \text{ double Higgs fusion:} & qq & \to & qq + WW/ZZ & \to & HH & \\ \text{associated production:} & pp & \to & t\bar{t}HH & & & \end{array} \quad (6.32)$$

The dominant gluon fusion production process proceeds via triangle and box diagrams, see Fig. 6.16.

Due to smallness of the cross sections, *cf.* Fig. 6.17, and the large QCD background the extraction of the Higgs self-coupling at the LHC is extremely difficult. There is an enormous

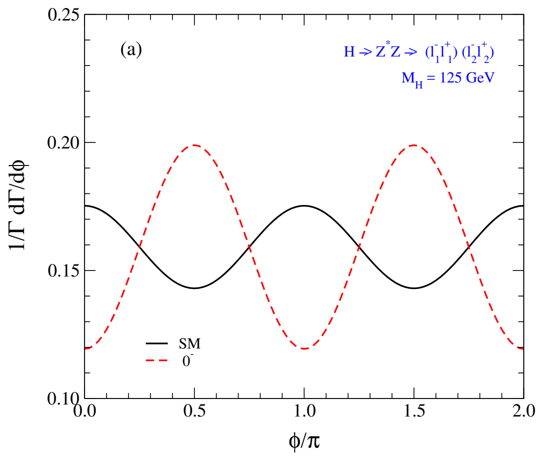


Abbildung 6.12: The azimuthal distribtuion for the $H \to ZZ^* \to 4l$ decay for the SM scalar Higgs (black) and a pseudoscalar (red). [Choi,Mühlleitner,Zerwas]

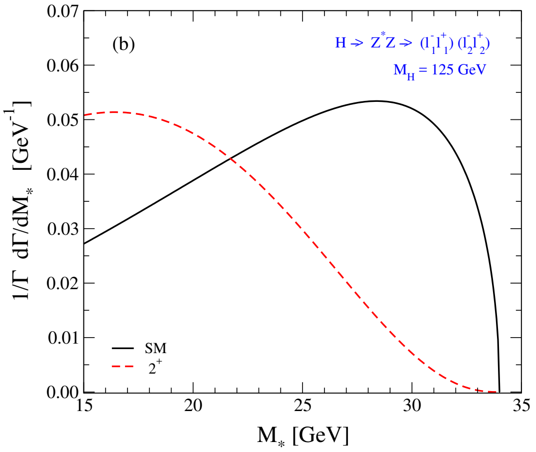


Abbildung 6.13: The threshold distribution for the $H \to ZZ^* \to 4l$ decay for the SM spin-0 Higgs (black) and a spin-2 particle (red).[Choi,Mühlleitner,Zerwas]

theoretical activity to determine the production processes with high accuracy including HO corrections and to develop strategies and observables for the measurement of the di-Higgs production processes and the trilinear Higgs self-couplings.

6.10 Summary

The measurements of the properties of the discovered particle have identified it as the Higgs boson. CERN therefore officially announced in a press release of March 2013, that the discovered particle is the Higgs boson, *cf.* Fig. 6.18. This lead then to the Nobel Prize for Physics in 2013 to Francois Englert and Peter Higgs.

The SM of particle physics has been very successful so far. At the experiments it has been tested to highest accuracy, including higher order corrections. And with the discovery of the Higgs particle we have found the last missing piece of the SM of particle physics. Still there are many open questions that cannot be answered by the SM. To name a few of them

1. In the SM the Higgs mechanism is introduced ad hoc. There is no dynamical mechanism behind it.

2. In the presence of high energy scales, the Higgs boson mass receives large quantum corrections, inducing the hierarchy problem.

3. We have no explanation for the fermion masses and mixings.

4. The SM does not contain a Dark Matter candidate.

5. In the SM the gauge couplings do not unify.

6. The SM does not incorporate gravity.

7. The CP violation in the SM is not large enough to allow for baryogenesis.

8. ...

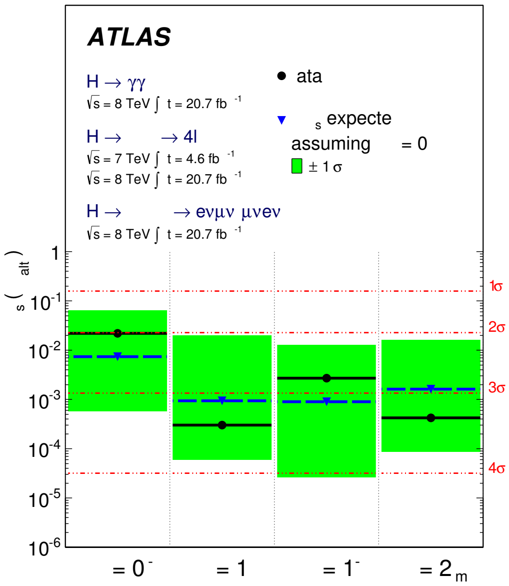


Abbildung 6.14: Spin-parity hypotheses tests by ATLAS. Details in Phys. Lett. B726 (2013) 120.

We therefore should rather see the SM as an effective low-energy theory which is embedded in some more fundamental theory that becomes apparent at higher scales. The Higgs data so far, although pointing towards a SM Higgs boson, still allow for interpretations within theories beyond the SM. These BSM theories ideally solve (some of) the problems of the SM. They are subject of further lectures.

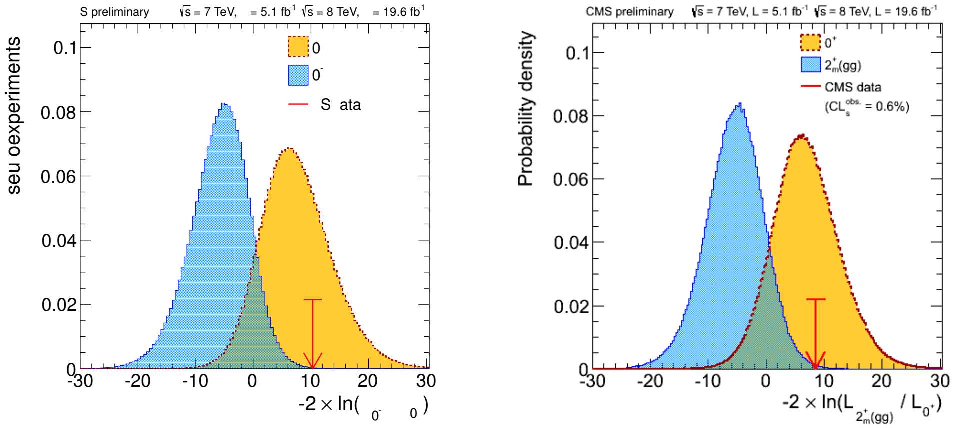


Abbildung 6.15: Spin-parity hypotheses tests by CMS. Left: 0^- excluded at 95% C.L. [CMS-PAS-HIG-13-002]. Right: $2^+_m(gg)$ excluded at 60% C.L. [CMS-PAS-HIG-13-005].

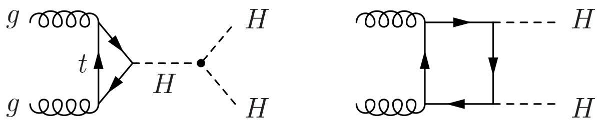


Abbildung 6.16: The diagrams which contribute to the gluon gluon fusion process $gg \to HH$.

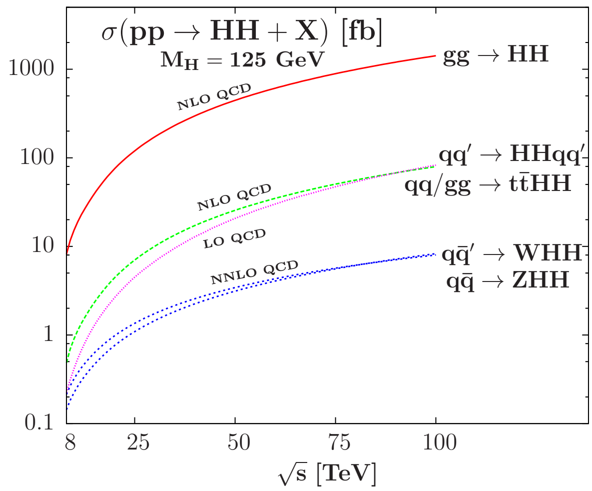
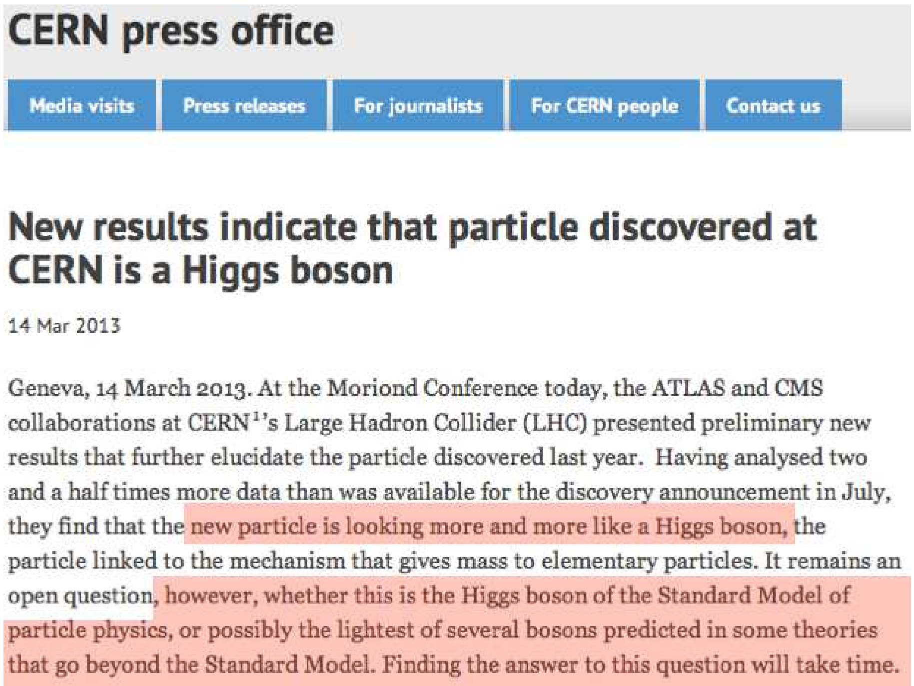


Abbildung 6.17: Di-Higgs production processes at the LHC with c.m. energy 14 TeV, including HO corrections. [Baglio,Djouadi,Gröber,Mühlleitner,Quévillon,Spira].



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New results indicate that particle discovered at CERN is a Higgs boson

14 Mar 2013

Geneva, 14 March 2013. At the Moriond Conference today, the ATLAS and CMS collaborations at CERN[1]'s Large Hadron Collider (LHC) presented preliminary new results that further elucidate the particle discovered last year. Having analysed two and a half times more data than was available for the discovery announcement in July, they find that the new particle is looking more and more like a Higgs boson, the particle linked to the mechanism that gives mass to elementary particles. It remains an open question, however, whether this is the Higgs boson of the Standard Model of particle physics, or possibly the lightest of several bosons predicted in some theories that go beyond the Standard Model. Finding the answer to this question will take time.

Abbildung 6.18: CERN press release.

Kapitel 7

Appendix

7.1 Addendum: Mathematische Hintergrundinformationen

7.1.1 Gruppen

Sei ein Paar $(G, *)$ mit einer Menge G und einer inneren zweistelligen Verküpfung/Gruppenmultiplikation. $* : G \times G \to G, (a, b) \mapsto a * b$ heißt Gruppe, wenn folgende Axiome erfüllt sind

1. Die Gruppe ist *abgeschlossen*. D.h. wenn $g, h \,\epsilon\, G \Rightarrow g * h \,\epsilon\, G$.
2. *Assoziativität:* $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$.
3. $\exists$ *Einselement* e mit der Eigenschaft $g * e = e * g = g \;\; \forall\, g \,\epsilon\, G$.
4. Zu jedem g gibt es ein *Inverses* g^{-1} mit $g^{-1} * g = g * g^{-1} = e$.

Abelsche Gruppe: Eine Gruppe heißt *abelsch*, wenn $g * h = h * g$.

Kontinuierliche Gruppen: Sie besitzen unendlich viele Elemente und werden durch n Parameter beschrieben. Bei *Liegruppen* ist n endlich. Alle einparametrigen Liegruppen sind abelsch. Typisches Beispiel: $U(1)$ mit den Elementen $e^{i\phi}$ und ϕ als Parameter.

7.1.2 Algebra

Ein linearer Raum (Vektorraum) wird zu einer *Algebra* $\mathbf{A}$, wenn eine binäre Operation (Multiplikation) zweier Elemente m, n existiert, so daß $mn \;\; \in \mathbf{A}$. Es gelten die Linearitätsbeziehungen $(k, m, n \;\in\; \mathbf{A})$

$$\begin{aligned} k(c_1 m + c_2 n) &= c_1 km + c_2 kn \\ (c_1 m + c_2 n)k &= c_1 mk + c_2 nk \,. \end{aligned} \tag{7.0}$$

Dabei sind c_1, c_2 reelle (komplexe) Zahlen. Man spricht je nach Fall von reeller (komplexer) Algebra.

Eine Algebra heißt *kommutativ*, wenn

$$mn = nm \,. \tag{7.1}$$

Sie heißt *assoziativ*, wenn

$$k(mn) = (km)n \; . \tag{7.2}$$

Sie heißt *Algebra mit Einselement*, wenn sie ein Einselement **1** besitzt mit

$$\mathbf{1}m = m\mathbf{1} = m \; . \tag{7.3}$$

Sei $\mathbf{A}$ eine *assoziative* Algebra mit Einselement und $B \subset \mathbf{A}$ eine Menge von Elementen b^1, b^2 etc. Die Algebra heißt von B *erzeugt*, wenn jedes $m \epsilon \mathbf{A}$ durch ein Polynom endlichen Grades in den Elementen b^i geschrieben werden kann,

$$m = c\mathbf{1} + \sum_{k=1}^{p} \sum_{i_1, i_2, ..., i_k} c_{i_1 i_2 ... i_k} b^{i_1} b^{i_2} ... b^{i_k} \; , \tag{7.4}$$

wobei die Koeffizienten $c_{i_1 i_2 ... i_k}$ komplexe Zahlen sind. Die Elemente der Menge B heißen *Generatoren* von $\mathbf{A}$. Das Einselement gehört nicht zu den Generatoren.

7.1.3 Clifford-Algebren

Eine Clifford-Algebra C_N wird von N Generatoren $\xi^1, \xi^2, ..., \xi^N$ erzeugt, für die

$$\boxed{\xi^a \xi^b + \xi^b \xi^a = 2\delta^{ab}}$$

mit $a, b = 1, ..., N$.

Die Dimension der Clifford-Algebra C_N ist 2^N. Es existiert ein enger Zusammenhang zwischen Clifford-Algebren und den Quantisierungsbedingungen für Fermionen.

Im allgemeinen lassen sich Clifford-Algebren für beliebige *symmetrische* Metriken g^{mn} definieren. So gilt insbesondere für die pseudoeuklidische Metrik

$$g_{ab} = \mathrm{diag}(\underbrace{1, 1, ..., 1}_{N}, \underbrace{-1, ..., -1}_{M}) \; , \tag{7.5}$$

Clifford-Algebra $C_{N,M}$: $\{\Gamma^m, \Gamma^n\} = 2g^{mn}\mathbf{1}$.

Die Anzahl der Generatoren ist $d = N + M$.

7.1.4 Liealgebren

Eine Algebra ist ein Vektorraum, der von den Generatoren $A, B, C, ...$ aufgespannt wird: beliebige Linearkombinationen von Generatoren ergeben wieder Generatoren. Eine Algebra verfügt über ein *Produkt* zwischen den Generatoren. Im Fall der Liealgebra ist das Produkt der Kommutator

$$A \circ B := [A, B] \; , \tag{7.6}$$

mit den folgenden Eigenschaften

$$A \circ B = -B \circ A \tag{7.7}$$

$$(A \circ B) \circ C + (C \circ A) \circ B + (B \circ C) \circ A = 0 \; . \tag{7.8}$$

Liealgebren sind nicht assoziativ. Die Beziehung (7.8) heißt *Jacobi-Identität*.